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(PDF says "version 2" at top.)

More Number Theory

CSE 311 24Au
Lecture 11

Extra Vacuous Truth resources

TAs noticed many errors with vacuous truth in the last homework.

Vacuous truth refers to only "$F \rightarrow ?$" lines of the implication truth table

"$T \rightarrow T$" is a true implication, but it's not vacuous.

"$T \rightarrow F$" is a false implication

If you don't see an implication with a false hypothesis it isn't vacuous truth.

Reading with examples and "why is vacuous truth the right definition" https://courses.cs.washington.edu/courses/cse311/24au/resources/reading02-vacuous.pdf

You Try!

Claim: for all integers a, b, c, n with $n > 0$:
If $a \equiv b \ (mod\ n)$ then $ac \equiv bc \ (mod\ n)$

Before we start we must know:

1. What every word in the statement means.
2. What the statement as a whole means.
3. Where to start.
4. What your target is.

Divides

For integers x, y we say $x|y$ ("x divides y") iff there is an integer z such that $xz = y$.

Equivalence in modular arithmetic

Let $a \in \mathbb{Z}, b \in \mathbb{Z}, n \in \mathbb{Z}$ and $n > 0$.
We say $a \equiv b \ (mod\ n)$ if and only if $n|(b - a)$

Claim: for all integers a, b, c, n, with $n > 0$:

$$a \equiv b \pmod{n} \rightarrow ac \equiv bc \pmod{n}$$

Proof:

Let a, b, c, n be arbitrary integers with $n > 0$ and suppose $a \equiv b(mod\ n)$.

$n \mid (b - a)$

① $nk = b - a$ (k int)

② $n \_\_\_ = bc - ac$

$n \mid bc - ac$

$ac \equiv bc\ (mod\ n)$

Claim: for all integers $a, b, c, n,$ with $n > 0$:

$$a \equiv b \pmod{n} \rightarrow ac \equiv bc \pmod{n}$$

Proof:

Let a, b, c, n be arbitrary integers with $n > 0$
and suppose $a \equiv b(mod\ n)$.

By definition of mod $n|(b - a)$

By definition of divides, $nk = b - a$ for some integer k

Multiplying both sides by c, we have $n(ck) = bc - ac$.

Since c and k are integers, $n|(bc - ac)$ by definition of divides.

So, $ac \equiv bc\ (mod\ n)$, by the definition of mod.

Don’t lose your intuition!

Let’s check that we understand “intuitively” what mod means:

$x \equiv 0 \pmod 2$

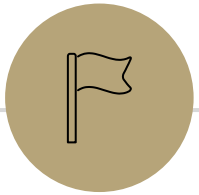
“x is even” Note that negative (even) x values also make this true.

$-1 \equiv 19 \pmod 5$

This is true! They both have remainder 4 when divided by 5.

$y \equiv 2 \pmod 7$

This is true as long as $y = 2 + 7k$ for some integer k



Proof by Contrapositive

Another Proof

For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

Another Proof

For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

Proof:

Let a, b, c be arbitrary integers, and suppose $a \nmid (bc)$.

Then there is not an integer z such that $az = bc$

…

So $a \nmid b$ or $a \nmid c$

Another Proof

For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

Proof:

Let a, b, c be arbitrar c).

Then there is not an

…

so $a \nmid b$ or $a \nmid c$

There has to be a better way!

Another Proof

For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

There has to be a better way!

If only there were some equivalent implication…

One where we could negate everything…

Take the contrapositive of the statement:

For all integers, a, b, c: Show if $a|b$ and $a|c$ then $a|(bc)$.

By contrapositive

Claim: For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

We argue by contrapositive.

Let a, b, c be arbitrary integers, and suppose $a|b$ and $a|c$.

Therefore $a|bc$

By contrapositive

Claim: For all integers, a, b, c: Show that if $a \nmid (bc)$ then $a \nmid b$ or $a \nmid c$.

We argue by contrapositive.

Let a, b, c be arbitrary integers, and suppose $a|b$ and $a|c$.

By definition of divides, $ax = b$ and $ay = c$ for integers x and y.

Multiplying the two equations, we get $axay = bc$

Since a, x, y are all integers, xay is an integer. Applying the definition of divides, we have $a|bc$.

Signs you might want to use proof by contrapositive

1. The hypothesis of the implication you're proving has a "not" in it (that you think is making things difficult)

2. The target of the implication you're proving has an "or" or "not" in it.

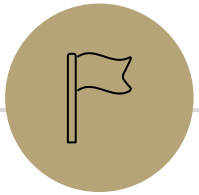
3. There's a step that is difficult forward, but easy backwards

   e.g., taking a square-root forward, squaring backwards.

4. You get halfway through the proof and you can't "get ahold of" what you're trying to show.

   e.g., you're working with a "not equal" instead of an "equals" or "every thing doesn't have this property" instead of "some thing does have that property"

All of these are reasons you **might** want contrapositive. Sometimes you just have to try and see what happens!



Logical Ordering

Logical Ordering

When doing a proof, we often work from both sides…

But we have to be careful!

When you read from top to bottom, every step has to follow only from what's **before** it, not after it.

Suppose our target is q and I know $q \to p$ and $r \to q$.

What can I put as a "new target?"

Logical Ordering

So why have all our prior steps been ok backward?

They've all been either:

A definition (which is always an "if and only if")

An algebra step that is an "if and only if"

Even if your steps are "if and only if" you still have to put everything in order – start from your assumptions, and only assert something once it can be shown.

A bad proof

Claim: if x is positive then $x + 5 = -x - 5$.

$x + 5 = -x - 5$

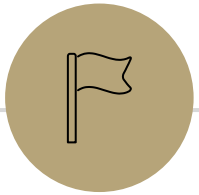
$|x + 5| = |-x - 5|$

$|x + 5| = |-(x + 5)|$

$|x + 5| = |x + 5|$

$0 = 0$

This claim is **false** – if you're trying to do algebra, you need to start with an equation you know (say $x = x$ or $2 = 2$ or $0 = 0$) and expand to the equation you want.



GCD and LCM

GCD and LCM

Greatest Common Divisor

The Greatest Common Divisor of a and b (gcd(a,b)) is the largest integer c such that $c|a$ and $c|b$

Least Common Multiple

The Least Common Multiple of a and b (lcm(a,b)) is the smallest positive integer c such that $a|c$ and $b|c$.

Try a few values…

gcd(100,125)

gcd(17,49)

gcd(17,34)

gcd(13,0)

lcm(7,11)

lcm(6,10)

How do you calculate a gcd?

You could:

Find the prime factorization of each

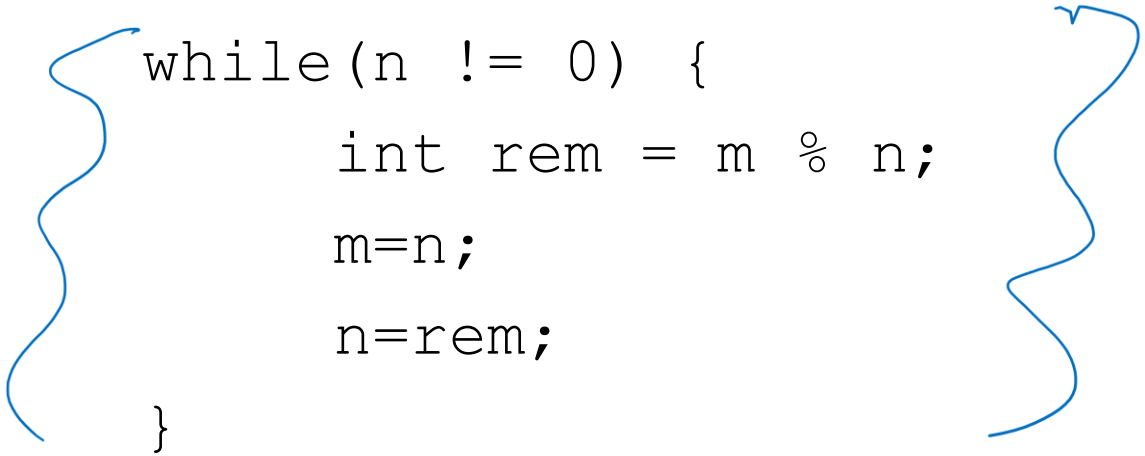
Take all the common ones. E.g.

gcd(24,20)=gcd($2^3 \cdot 3, 2^2 \cdot 5$) = 2^{min(2,3)} = 2^2 = 4.

(lcm has a similar algorithm – take the maximum number of copies of everything)

But that's….really expensive. Mystery finds gcd.

```
public int Mystery(int m, int n){
      if(m<n){
            int temp = m;
            m=n;
            n=temp;
      }
      while(n != 0) {
            int rem = m % n;
            m=n;
            n=rem;
      }
      return m;
}
```



Running Mystery

gcd(26,7) = gcd(7, 26%7) = gcd(7,5)

= gcd(5, 7%5) = gcd(5,2)

= gcd(2, 5%2) = gcd(2, 1)

= gcd(1, 2%1) = gcd(1,0)= 1.

GCD facts

1. gcd(a,0)=a

Pf: a is a common divisor ($a = 1 \cdot a$; $0 = 0 \cdot a$); larger numbers don't divide a (for positive numbers, if $x|y$ then $x \leq y$)

2. If a and b are positive integers, then gcd(a,b) = gcd(b, a % b)

Why is 2 true? The proof isn't easy, it's at the end of this deck.

Why should you care?

So…what's it good for?

Remember everything we're learning contributes to us eventually understanding RSA. This is a key step in generating keys.

Suppose I want to solve $7x \equiv 3 (mod\ n)$

Just multiply both sides by $\frac{1}{7}$…

Oh wait. We want a number to multiply by 7 to get 1.

What number can we pick?

The next two slides are going to get more abstract…we're listing out the facts we need to solve that equation.

Bézout's Theorem

Bézout's Theorem

If a and b are positive integers, then there exist integers s and t such that
gcd(a,b)= $sa + tb$

We're not going to prove this theorem…

But it turns out `Mystery` can be extended to find them.

Finding the inverse…

gcd(26,7) = gcd(7, 26%7) = gcd(7,5)

= gcd(5, 7%5) = gcd(5,2)

= gcd(2, 5%2) = gcd(2, 1)

= gcd(1, 2%1) = gcd(1,0)= 1.

$$26 = 3 \cdot 7 + 5 \; ; \; 5 = 26 - 3 \cdot 7$$

$$7 = 5 \cdot 1 + 2 \; ; \; 2 = 7 - 5 \cdot 1$$

$$5 = 2 \cdot 2 + 1 \; ; \; 1 = 5 - 2 \cdot 2$$

$$\begin{gathered} 1 = 5 - 2 \cdot 2 \\ = 5 - 2(7 - 5 \cdot 1) \\ = 3 \cdot 5 - 2 \cdot 7 \\ = 3 \cdot (26 - 3 \cdot 7) - 2 \cdot 7 \\ 3 \cdot 26 - 11 \cdot 7 \end{gathered}$$

-11 is a multiplicative inverse of 7 for (mod 26) arithmetic!
We'll write that as 15, since we're working mod 26.

So…what's it good for?

Suppose I want to solve $7x \equiv 3 (mod\ n)$

Just multiply both sides by $\frac{1}{7}$…

Oh wait. We want a number to multiply by 7 to get 1.

If the gcd(7,n) = 1

Then $s \cdot 7 + tn = 1$, so $7s - 1 = -tn$ i.e. $n|(7s - 1)$ so $7s \equiv 1 (mod\ n)$.

So the s from Bézout's Theorem is what we should multiply by!

Ok…how am I supposed to find s, t?

It turns out that while you're calculating the gcd (using the Mystery algorithm), you can keep some extra information recorded, and end up with the s, t

This is called the "extended Euclidian algorithm"

Examples in these slides.

Try it

Solve the equation $7y \equiv 3(mod\ 26)$

What do we need to find?

The multiplicative inverse of $7(\mathbf{mod}\ 26)$

Finding the inverse…

gcd(26,7) = gcd(7, 26%7) = gcd(7,5)

= gcd(5, 7%5) = gcd(5,2)

= gcd(2, 5%2) = gcd(2, 1)

= gcd(1, 2%1) = gcd(1,0)= 1.

$$26 = 3 \cdot 7 + 5 \; ; \; 5 = 26 - 3 \cdot 7$$

$$7 = 5 \cdot 1 + 2 \; ; \; 2 = 7 - 5 \cdot 1$$

$$5 = 2 \cdot 2 + 1 \; ; \; 1 = 5 - 2 \cdot 2$$

$$\begin{gathered} 1 = 5 - 2 \cdot 2 \\ = 5 - 2(7 - 5 \cdot 1) \\ = 3 \cdot 5 - 2 \cdot 7 \\ = 3 \cdot (26 - 3 \cdot 7) - 2 \cdot 7 \\ 3 \cdot 26 - 11 \cdot 7 \end{gathered}$$

-11 is a multiplicative inverse of 7 for (mod 26) arithmetic!
We'll write that as 15, since we're working mod 26.

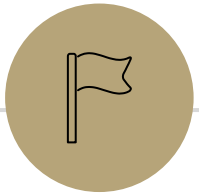
Try it

Solve the equation $7y \equiv 3(mod\ 26)$

What do we need to find?
The multiplicative inverse of $7\ (mod\ 26)$. We found it's 15.

$15 \cdot 7 \cdot y \equiv 15 \cdot 3(mod\ 26)$

$y \equiv 45(mod\ 26)$
Or $y \equiv 19(mod\ 26)$
So $26|19 - y$, i.e. $26k = 19 - y$ (for $k \in \mathbb{Z}$) i.e. $y = 19 - 26 \cdot k$ for any $k \in \mathbb{Z}$
Solutions: $\{\dots, -7, 19, 45, \dots 19 + 26k, \dots\}$ i.e. $\{x: x = 19 + 26k$ for some $k \in \mathbb{Z}\}$



Proving the key fact about gcds

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that y is a common divisor of a and b.

By definition of gcd, $y|b$ and $y|(a\%b)$. So it is enough to show that $y|a$.

Applying the definition of divides we get $b = yk$ for an integer k, and $(a\%b) = yj$ for an integer j.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q.

Plugging in both of our other equations:

$a = qyk + yj = y(qk + j)$. Since q, k, and j are integers, $y|a$. Thus y is a common divisor of a, b and thus $y \leq x$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

By definition of gcd, $\mathrm{x}|b$ and $x|a$. So it is enough to show that $\mathrm{x}|(a\%b)$.

Applying the definition of divides we get $b = xk'$ for an integer k', and $\mathrm{a} = xj'$ for an integer j'.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q

Plugging in both of our other equations:

$xj' = qxk' + a\%b$. Solving for $a\%b$, we have $a\%b = xj' - qxk' = x(j' - qk')$. So $x|(a\%b)$. Thus x is a common divisor of $b, a\%b$ and thus $x \leq y$.

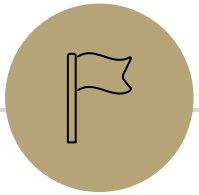
gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

We have shown $x \leq y$ and $y \leq x$.

Thus $x = y$, and $\gcd(a, b) = \gcd(b, a\%b)$.



More Mod proofs

More proofs

Show that if $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

Step 1: What do the words mean?

Step 2: What does the statement as a whole say?

Step 3: Where do we start?

Step 4: What's our target?

Step 5: Now prove it.

Another Proof

Show that if $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

Let $a, b, c, d, n \in \mathbb{Z}, n \geq 0$
and suppose $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$.

$ac \equiv bd(mod\ n)$

Another Proof

Show that if $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

Let $a, b, c, d, n \in \mathbb{Z}, n \geq 0$
and suppose $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$.

$n|(b-a)$ and $n|(d-c)$ by definition of mod.

$nk = (b-a)$ and $nj = (d-c)$ for integers j, k by definition of divides.

$n?? = bd - ac$

$n|(bd - ac)$

$ac \equiv bd(mod\ n)$

Another Proof

Show that if $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

Let $a, b, c, d, n \in \mathbb{Z}, n \geq 0$
and suppose $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$.

$n|(b-a)$ and $n|(d-c)$ by definition of mod.

$nk = (b-a)$ and $nj = (d-c)$ for integers j, k by definition of divides.

$nknj = (d-c)(b-a)$ by multiplying the two equations

$nknj = (bd - bc - ad + ac)$

…

$n?? = bd - ac$

$n|(bd-ac)$

$ac \equiv bd(mod\ n)$

Another Proof

Show that if $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

Let $a, b, c, d, n \in \mathbb{Z}, n \geq 0$
and suppose $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$.

$n|(b-a)$ and $n|(d-c)$ by definition of mod.

$nk = (b-a)$ and $nj = (d-c)$ for integers j, k by definition of divides.

$nknj = (d-c)(b-a)$ by multiplying the two equations

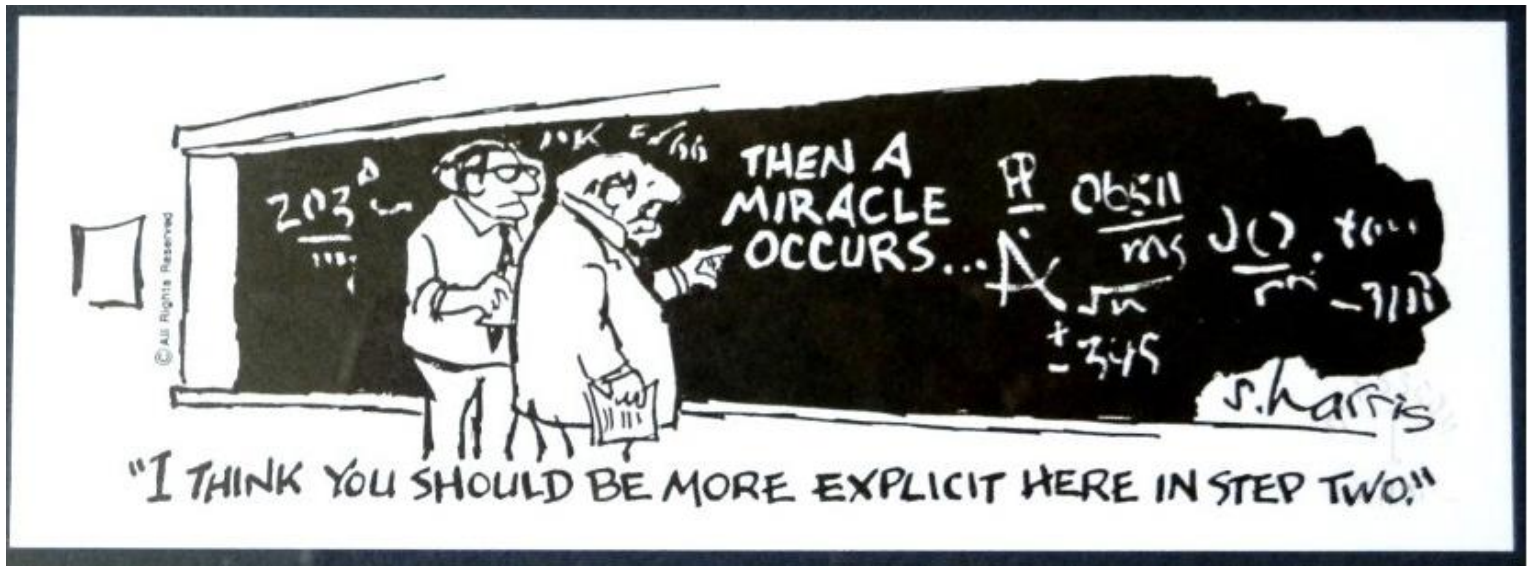
$nknj = (bd - bc - ad + ac)$

And then a miracle occurs

$n?? = bd - ac$

$n|(bd - ac)$

$ac \equiv bd(mod\ n)$



Uh-Oh

We hit (what looks like) a dead end.

But how did I know we hit a dead end? Because I knew exactly where we needed to go. If you didn't, you'd have been staring at that for ages trying to figure out the magic step.

(or worse, assumed you lost a minus sign somewhere, and just "fixed" it….)

Let's try again. This time, let's **separate** b from a and d from c before combining.

Another Approach

Show that if $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

Let $a, b, c, d, n \in \mathbb{Z}, n \geq 0$
and suppose $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$.

$n|(b-a)$ and $n|(d-c)$ by definition of mod.

$nk = (b-a)$ and $nj = (d-c)$ for integers j, k by definition of divides.

$b = nk + a, d = nj + c$

$n?? = bd - ac$

$n|(bd - ac)$

$ac \equiv bd(mod\ n)$

Another Approach

Show that if $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

Let $a, b, c, d, n \in \mathbb{Z}, n \geq 0$
and suppose $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$.

$n|(b-a)$ and $n|(d-c)$ by definition of mod.

$nk = (b-a)$ and $nj = (d-c)$ for integers j, k by definition of divides.

$b = nk + a, d = nj + c$,

$bd = (nk + a)(nj + c) = n^2kj + anj + cnk + ac$

$bd - ac = n^2kj + anj + cnk = n(nkj + aj + ck)$

$n?? = bd - ac$

$n|(bd - ac)$

$ac \equiv bd(mod\ n)$

Another Approach

Show that if $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$ then $ac \equiv bd\ (mod\ n)$.

Let $a, b, c, d, n \in \mathbb{Z}, n \geq 0$
and suppose $a \equiv b(mod\ n)$ and $c \equiv d\ (mod\ n)$.

$n|(b-a)$ and $n|(d-c)$ by definition of mod.

$nk = (b-a)$ and $nj = (d-c)$ for integers j, k by definition of divides.

Isolating,b and d, we have: $b = nk + a, d = nj + c$

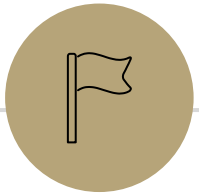
Multiplying the equations, and factoring, $bd = (nk + a)(nj + c) = n^2kj + anj + cnk + ac$

Rearranging, and facoring out n: $bd - ac = n^2kj + anj + cnk = n(nkj + aj + ck)$

Since all of $n, j, k, a,$ and c are integers, we have that $bd - ac$ is n times an integer, so

$n|(bd - ac)$, and by definition of mod

$ac \equiv bd(mod\ n)$



Euclidian Algorithm

Euclid's Algorithm

```
while(n != 0) {
        int rem = m % n;
        m=n;
        n=rem;
    }
```

gcd(660,126)

Euclid's Algorithm

```
while(n != 0) {
        int rem = m % n;
        m=n;
        n=rem;
}
```

gcd(660,126) = gcd(126, 660 mod 126) = gcd(126, 30)
= gcd(30, 126 mod 30) = gcd(30, 6)
= gcd(6, 30 mod 6) = gcd(6, 0)
= 6

Tableau form

$$660 = 5 \cdot 126 + 30$$
$$126 = 4 \cdot 30 + 6$$
$$30 = 5 \cdot 6 + 0$$

Starting Numbers

Final answer

Bézout's Theorem

Bézout's Theorem

If a and b are positive integers, then there exist integers s and t such that
gcd(a,b)= $sa + tb$

We're not going to prove this theorem…

But we'll show you how to find s,t for any positive integers a, b.

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

gcd(35,27)

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

gcd(35,27) = gcd(27, 35%27) = gcd(27,8)
= gcd(8, 27%8) = gcd(8, 3)
= gcd(3, 8%3) = gcd(3, 2)
= gcd(2, 3%2) = gcd(2,1)
= gcd(1, 2%1) = gcd(1,0)

$$35 = 1 \cdot 27 + 8$$
$$27 = 3 \cdot 8 + 3$$
$$8 = 2 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$
\begin{aligned}
35 &= 1 \cdot 27 + 8 \\
27 &= 3 \cdot 8 + 3 \\
8 &= 2 \cdot 3 + 2 \\
3 &= 1 \cdot 2 + 1
\end{aligned}
$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$\begin{aligned} 35 &= 1 \cdot 27 + 8 \\ 27 &= 3 \cdot 8 + 3 \\ 8 &= 2 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \end{aligned}$$

$$\begin{aligned} 8 &= 35 - 1 \cdot 27 \\ 3 &= 27 - 3 \cdot 8 \\ 2 &= 8 - 2 \cdot 3 \\ 1 &= 3 - 1 \cdot 2 \end{aligned}$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$
\begin{aligned}
8 &= 35 - 1 \cdot 27 \\
3 &= 27 - 3 \cdot 8 \\
2 &= 8 - 2 \cdot 3 \\
1 &= 3 - 1 \cdot 2
\end{aligned}
$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$
\begin{aligned}
8 &= 35 - 1 \cdot 27 \\
3 &= 27 - 3 \cdot 8 \\
2 &= 8 - 2 \cdot 3 \\
1 &= 3 - 1 \cdot 2
\end{aligned}
$$

$$
\begin{aligned}
1 &= 3 - 1 \cdot 2 \\
&= 3 - 1 \cdot (8 - 2 \cdot 3) \\
&= -1 \cdot 8 + 2 \cdot 3
\end{aligned}
$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$
\begin{aligned}
8 &= 35 - 1 \cdot 27 \\
3 &= 27 - 3 \cdot 8 \\
2 &= 8 - 2 \cdot 3 \\
1 &= 3 - 1 \cdot 2
\end{aligned}
$$

gcd(27,35) = $13 \cdot 27 + (-10) \cdot 35$

$$
\begin{aligned}
1 &= 3 - 1 \cdot 2 \\
&= 3 - 1 \cdot (8 - 2 \cdot 3) \\
&= -1 \cdot 8 + 3 \cdot 3 \\
&= -1 \cdot 8 + 3(27 - 3 \cdot 8) \\
&= 3 \cdot 27 - 10 \cdot 8 \\
&= 3 \cdot 27 - 10(35 - 1 \cdot 27) \\
&= 13 \cdot 27 - 10 \cdot 35
\end{aligned}
$$

When substituting back, you keep the larger of m, n and the number you just substituted. Don't simplify further! (or you lose the form you need)