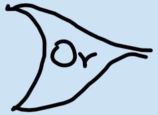
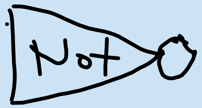
No I am not Robbie

Predicates and Quantifiers

CSE 311 Autumn 24
Lecture 5

Meet Boolean Algebra

Name	Variables	"True/False"	"And"	"Or"	"Not"	Implication
Java Code	`boolean b`	`true,false`	`&&`	`||`	`!`	No special symbol
Propositional Logic	"p, q, r"	`T, F`	$\wedge$	$\vee$	$\neg$	$\rightarrow$
Circuits	Wires	1, 0				No special symbol
Boolean Algebra	a, b, c	1,0	$\cdot$ ("multiplication")	$+$ ("addition")	$'$ (apostrophe after variable)	No special symbol

Propositional logic

$(p \wedge q \wedge r) \vee s \vee \neg t$

Boolean Algebra

$pqr + s + t'$

Predicates!

Predicate Logic

So far our propositions have worked great for fixed objects.

What if we want to say "If $x > 10$ then $x^2 > 100$."

$x > 10$ isn't a proposition. Its truth value depends on x.

We need a function that can take in a value for x and output True or False as appropriate.

Predicates

Predicate

A function that outputs true or false.

`Cat(x)` := "`x` is a cat"

`Prime(x)` := "`x` is prime"

`LessThan(x,y)` := "`x<y`"

`Sum(x,y,z)` := "`x+y=z`"

`HasNChars(s,n)` := "string `s` has length `n`"

Numbers and types of inputs can change. Only requirement is output is Boolean.

Analogy

Propositions were like Boolean variables.

What are predicates? Functions that return Booleans

```
public boolean predicate(…)
```

Translation

Translation works a lot like when we just had propositions.

Let's try it...

x is prime or x^2 is odd or $x = 2$.

$\texttt{Prime}(x) \vee \texttt{Odd}(x^2) \vee \texttt{Equals}(x, 2)$

Prime(a) := "a is prime"

Odd(a) := "a is Odd"

Is2(a) := "a is equal 2"

$\text{Prime}(x) \vee \text{Odd}(x^2) \vee \text{Is2}(x)$

Domain of Discourse

x is prime or x^2 is odd or $x = 2$.

$\texttt{Prime}(x) \vee \texttt{Odd}(x^2) \vee \texttt{Equals}(x, 2)$

Can x be 4.5? What about “abc” ?

I never intended you to plug 4.5 or “abc” into x.

When you read the sentence you probably didn’t imagine plugging those values in….

Domain of Discourse

x is prime or x^2 is odd or $x = 2$.

$\texttt{Prime}(x) \lor \texttt{Odd}(x^2) \lor \texttt{Equals}(x, 2)$

To make sure we **can't** plug in 4.5 for x, predicate logic requires deciding on the types we'll allow

Domain of Discourse

The set of all inputs allowed as inputs to our predicates.

Often we give the type(s) of allowed inputs, like "all integers" or "all real numbers.

Try it…

What's a possible domain of discourse for these lists of predicates?

1. "x is a cat", "x barks", "x likes to take walks"

   Set of all animals, mammals

2. "x is prime", "x=5" "$x < 20$" "x is a power of two"

   set of all integers, natural numbers

3. "x is enrolled in course y", "y is a pre-req for z"

Try it…

What's a possible domain of discourse for these lists of predicates?

1. "x is a cat", "x barks", "x likes to take walks"

   "Mammals", "pets", "dogs and cats", …

2. "x is prime", "x=5" "$x < 20$" "x is a power of two"

   "positive integers", "integers", "numbers", …

3. "x is enrolled in course y", "y is a pre-req for z"

   "objects in the university course enrollment system", "university entities", "students and courses", …

More than one domain of discourse might be reasonable…if it might affect the meaning of the statement, we specify it.

Quantifiers

Now that we have variables, let's really use them...

We tend to use variables for two reasons:

1. The statement is true for every x, we just want to put a name on it.
2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

Quantifiers

$\forall \to A$ all $\exists \to E \to$ exists

We have two extra symbols to indicate which way we're using the variable.

1. The statement is true for every x, we just want to put a name on it.

$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, such that $p(x)$ and $q(x)$ are both true.

Quantifiers

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Universal Quantifier

"$\forall x$"

"for each x", "for every x", "for all x" are common translations

Remember: upside-down-A for All.

Quantifiers

$\forall x \forall y$ $\forall x, y$

Existential Quantifier

"$\exists x$"

"there is an x", "there exists an x", "for some x" are common translations

Remember: backwards-E for Exists.

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, for which $p(x)$ and $q(x)$ are both true.

Translations

Domain of Discourse: set of all integers

Even(a) := a is even

Equals(a, b) := a is equal to b

"For every x, if x is even, then $x = 2$."

$\forall x\,(\text{Even}(x) \rightarrow \text{Equals}(x, 2))$

"There are x, y such that x $< y$."

$\exists x\,(\texttt{Odd}(x) \wedge \texttt{LessThan}(x, 5))$

$\forall y\,(\text{Even}(y) \wedge \text{Odd}(y))$

For all values of y, y is both even and odd.

Translations

"For every x, if x is even, then $x = 2$."

$\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

"There are x, y such that x $< y$."

$\exists x \exists y(\texttt{LessThan}(x, y))$

$\exists x, y$ $\exists (x, y)$ ( )

$\exists x\ (\texttt{Odd}(x) \wedge \texttt{LessThan}(x, 5))$

There is an odd number that is less than 5.

$\forall y\ (\text{Even}(y) \wedge \text{Odd}(y))$

All numbers are both even and odd.

Translations

More practice in section and on homework.

Also a reading on the webpage –

- An explanation of why "for any" is not a great way to translate ∀ (even though it looks like a good option on the surface)
- More information on what happens with multiple quantifiers (we'll discuss more on Monday).

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

TRICK QUESTION! It depends on the domain.

Prime Numbers	Positive Integers	Odd integers
True	False	True (vacuously)

One Technical Matter

How do we parse sentences with quantifiers?

What's the "order of operations?"

We will usually put parentheses right after the quantifier and variable to make it clear what's included. If we don't, it's the rest of the expression.

Be careful with repeated variables…they don't always mean what you think they mean.

$\forall x\big(P(x)\big) \wedge \forall x(Q(x))$ are different x's.

Bound Variables

What happens if we repeat a variable?

Whenever you introduce a new quantifier with an already existing variable, it "takes over" that name until its expression ends.

$$\forall x(P(x) \land \forall x[Q(x)] \land R(x))$$

It's common (albeit somewhat confusing) practice to reuse a variables when it "wouldn't matter".

Never do something like the above: where a single name switches from gold to purple back to gold. Switching from gold to purple only is usually fine…but names are cheap.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

For every fruit, if it is not ripe then it is not tasty.

There is a fruit that is sliced and diced.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

$$\exists x(\texttt{Tasty}(x) \wedge \texttt{Ripe}(x))$$

For every fruit, if it is not ripe then it is not tasty.

$$\forall x(\neg\texttt{Ripe}(x) \rightarrow \neg\texttt{Tasty}(x))$$

There is a fruit that is sliced and diced.

$$\exists x(\texttt{Sliced}(x) \wedge \texttt{Diced}(x))$$

Negation

Negating Quantifiers

What happens when we negate an expression with quantifiers?

What does your intuition say?

Original	Negation
Every positive integer is prime	There is a positive integer that is not prime.
$\forall x\ \texttt{Prime}(x)$ Domain of discourse: positive integers	$\exists x(\neg\ \texttt{Prime}(x))$ Domain of discourse: positive integers

Negating Quantifiers

Let's try on an existential quantifier…

Original	Negation
There is a positive integer which is prime and even.	Every positive integer is composite or odd.
$\exists x(\mathtt{Prime}(x) \wedge \mathtt{Even}(x))$	$\forall x(\neg\mathtt{Prime}(x) \vee \neg\mathtt{Even}(x))$
Domain of discourse: positive integers	Domain of discourse: positive integers

To negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

Negation

Let your Domain of Discourse be integers; translate into predicate notation and negate.

There are integers x, y such that $xy = 0$.

Every integer is even.

Negation

Let your Domain of Discourse be integers; translate into predicate notation and negate.

There are integers x, y such that $xy = 0$.

Original $\exists x \exists y(\text{Equal}(xy, 0))$

Negation $\forall x \forall y(\neg \text{Equal}(xy, 0))$

Every integer is even.

Original $\forall x(\text{Even}(x))$

Negation $\exists x(\neg \text{Even}(x))$