Warm Up

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

It is snowing today, and if it is raining or snowing then we won’t walk to school.

Warm Up – Solution

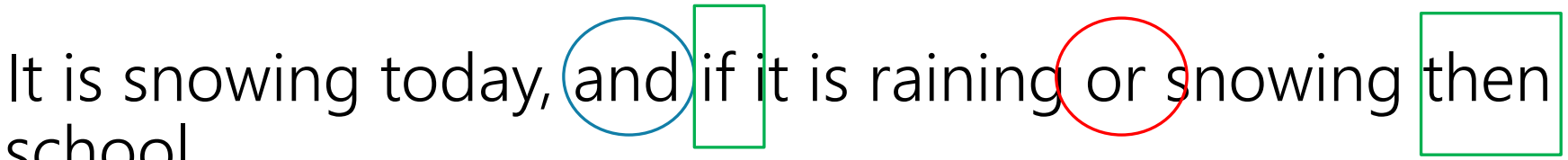
Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

It is snowing today, and if it is raining or snowing then we won't walk to school.

Robbie's process: identify connecting words, identify propositions, figure out parentheses.

Warm Up – Solution

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.



Identify connecting words: look for and, or, not, if-then, etc.

Warm Up – Solution

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

p q p $\neg r$

It is snowing today, and if it is raining or snowing then we won't walk to school.

Identify propositions: What's left are propositions, look for repeats and hidden negations.

p:it is snowing today.
q: it is raining.
r: we walk to school.

Warm Up – Solution

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

p q p $\neg r$

It is snowing today, and if it is raining or snowing then we won't walk to school.

p:it is snowing today.
q: it is raining.
r: we walk to school.

Figure out parentheses

$$p \wedge [(q \vee p) \rightarrow \neg r]$$

"raining or snowing" is the condition of the implication, not walking to school is the conclusion. Omitted words in other clauses hint that "It is snowing today" stands on its own.

xkcd.com/1153

Our First Proof and Digital Logic

CSE 311 Winter 2024
Lecture 3

Announcements: Office Hours

A chance to talk to staff about the class.

Advice:

You don't have to have a question to come to office hours! (It can help, though)

Aim early on in the "homework cycle" (i.e., long before the deadline).

We won't answer every kind of question in office hours:

We don't "pregrade" homework. We won't tell you whether something is right or wrong.

We will help on homework, but not usually by "giving hints." We'll usually ask questions about what you've tried, ask questions to help you decide what to do next, or point you to good examples to look at.

You're allowed to talk to others at office hours, as long as you're still following the collaboration policy.

Announcements: Section Materials

Handouts and solutions from sections are on the calendar.

Section solutions should be one of your most-used resources.

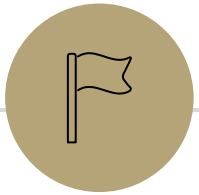
They're staff-written solutions for problems that are often similar to homeworks.

When you're wondering how much explanation to give, or what kind of formatting we might expect, section solutions are the first place to look (along with lecture slides).

Today

Our first proof!

Contrapositives and digital logic.



Simplifying Expressions

Logical Equivalence

We will want to talk about whether two propositions are "the same."

Two propositions are "equal" (=) if they are character-for-character identical.

$p \wedge q = p \wedge q$ but $p \wedge q \neq q \wedge p$

We almost never ask whether propositions are equal. It's not an interesting question.

Two propositions are "equivalent" ($\equiv$) if they always have the same truth value.

$p \wedge q \equiv p \wedge q$ and $p \wedge q \equiv q \wedge p$

But $p \wedge q \not\equiv p \vee q$
When p is true and q is false: $p \wedge q$ is false, but $p \vee q$ is true.

$A \leftrightarrow B$ vs. $A \equiv B$

A ≡ B is an ***assertion over all possible truth values*** that A and B always have the same truth values.

Use $A \equiv B$ when you're manipulating propositions ("doing algebra")

A ↔ B is a ***proposition*** that may be true or false depending on the truth values of the variables in A and B.

This distinction will be easier to understand after you see us use them both a few time.

Manipulating Expressions

When we're doing algebra, we can apply rules to transform expressions

$(a + b)(c + d) = ac + ad + bc + bd$ or $ab + ac = a(b + c)$

We want rules for logical expressions too.

For two rules, we'll:

1. Derive it/make sure we understand why it's true.
2. Practice using it.

By the end of the course, you'll do these "automatically" on full sentences; for now we'll practice mechanically on symbolic forms.

As you're practicing, don't lose sight of the *intuition* for what you're doing.

De Morgan's Laws

Negate the statement

"my code compiles or there is a bug."

i.e. find a natural English sentence that says $\neg(p \vee q)$

"the following is not true: my code compiles or there is a bug"

Hint: when it the original sentence false?

'or' means 'at least one is true' so to negate, we need to say 'neither is true' or equivalently 'both are false'

$\neg p \wedge \neg q$

"my code does not compile and there is not a bug"

De Morgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

is a general rule. It's always true for any propositions p and q.
This is one of De Morgan's Laws.

The other is:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

De Morgan's Laws

Example: $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Every line is the same!
So these expressions are equivalent.

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
if (!(front != null && value > front.data))
      front = new ListNode(value, front);
else {
      ListNode current = front;
      while (current.next != null && current.next.data < value))
            current = current.next;
      current.next = new ListNode(value, current.next);
}
```

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
!(front != null && value > front.data)
```

≡

```
front == null || value <= front.data
```

You've been using these for a while!

Law of Implication

Implications are hard.
AND/OR/NOT make more intuitive sense to me…
can we rewrite implications using just ANDs ORs and NOTs?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

One approach: think "when is this implication false?" then negate it (you might want one of DeMorgan's Laws!

Law of Implication

Implications are hard.
AND/OR/NOT make more intuitive sense to me…
can we rewrite implications using just ANDs ORs and NOTs?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	T	T

Seems like we might want $\neg(p \wedge \neg q)$

$\neg p \vee q$

Seems like a reasonable guess.
So is it true? Is $\neg p \vee q \equiv p \rightarrow q$?

Law of Implication

$$\neg p \vee q \equiv p \rightarrow q$$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Every line is the same!
So these expressions are equivalent.

Properties of Logical Connectives

We've derived two facts about logical connectives.

There's a lot more. A LOT more.

The next slide is a list of a bunch of them…

- Most of these are much less complicated than the last two, so we won't go through them in detail.
- DO NOT freak out about how many there are. We will always provide you the list on the next slide (no need to memorize).

Properties of Logical Connectives

These identities hold for all propositions p, q, r

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Using Our Rules

WOW that was a lot of rules.

Why do we need them? Simplification!

Let's go back to the "law of implication" example.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	T	T

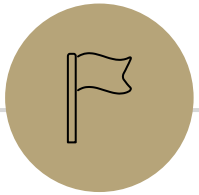
When is the implication true? Just "or" each of the three "true" lines!

$$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$

Also seems pretty reasonable

So is $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv (\neg p \vee q)$

i.e. are these both alternative representations of $p \rightarrow q$?



Our First Proof

Our First Proof

We could make another truth table (you should! It's a good exercise)

But we have another technique that is nicer.

Let's try that one

Then talk about why it's another good option.

We're going to give an iron-clad guarantee that:

$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv \neg p \vee q$

i.e. that this is another valid "law of implication"

Our First Proof

This will be a long proof! Longer than most of the ones on homeworks.

I'm starting with a hard one so you see all the tricks.

This process will be easier if we change variables, we're going to show

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv \neg a \vee b$

How do we write a proof?

It's not always plug-and-chug…we'll be highlighting strategies throughout the quarter.

To start with:

Make sure we know what we want to show…

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv$

None of the rules look like this

Practice of Proof-Writing:
Big Picture…WHY do we think this might be true?

The last two "pieces" came from the vacuous proof lines…maybe the "$\neg a$" came from there? Maybe that **simplifies** down to $\neg a$

$\equiv (\neg a \vee b)$

Let’s apply a rule

$(\neg a \wedge b) \vee (\neg a \wedge \neg b)$

The law says:

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$(\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv \neg a \wedge (b \vee \neg b)$

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv$

None of the rules look like this

Practice of Proof-Writing:
Big Picture…WHY do we think this might be true?

The last two "pieces" came from the vacuous proof lines…maybe the "$\neg a$" $\equiv (\neg a \vee b)$ came from there? Maybe that **simplifies** down to $\neg a$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Set ourselves an intermediate goal.
Let's try to simplify those last two pieces

Associative law
Connect up the things we're working on.

$$\equiv (\neg a \vee b)$$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$

Set ourselves an intermediate goal.
Let's try to simplify those last two pieces

Distributive law
We think $\neg a$ is important, let's isolate it.

$$\equiv (\neg a \vee b)$$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$$

Set ourselves an intermediate goal.
Let's try to simplify those last two pieces

Negation
Should make things simpler.

$$\equiv (\neg a \vee b)$$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$$
$$\equiv (a \wedge b) \vee [\neg a]$$

Set ourselves an intermediate goal. Let's try to simplify those last two pieces

Identity
Should make things simpler.

$$\equiv (\neg a \vee b)$$

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$

$\equiv (a \wedge b) \vee [\neg a]$

Stay on target:

We met our intermediate goal.

Don't forget the final goal!

We want to end up at $(\neg a \vee b)$

If we apply the distribution rule,

We'd get a $(\neg a \vee b)$

$\equiv (\neg a \vee b)$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Stay on target:
We met our intermediate goal.
Don't forget the final goal!
We want to end up at $(\neg a \vee b)$

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$$
$$\equiv (a \wedge b) \vee [\neg a]$$
$$\equiv [\neg a] \vee (a \wedge b)$$

If we apply the distribution rule,
We'd get a $(\neg a \vee b)$

Commutative
Make the expression look exactly like the law (more on this later)

$$\equiv (\neg a \vee b)$$

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Stay on target:
We met our intermediate goal.
Don't forget the final goal!
We want to end up at $(\neg a \vee b)$

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$
$$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$$
$$\equiv (a \wedge b) \vee [\neg a]$$
$$\equiv [\neg a] \vee (a \wedge b)$$

If we apply the distribution rule,
We'd get a $(\neg a \vee b)$

$$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$$

Distributive
Creates the $(\neg a \vee b)$ we were hoping for.

$$\equiv (\neg a \vee b)$$

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b)$

$\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$

$\equiv (a \wedge b) \vee [\neg a]$

$\equiv [\neg a] \vee (a \wedge b)$

$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$

$\equiv (a \vee \neg a) \wedge (\neg a \vee b)$

$\equiv \mathrm{T} \wedge (\neg a \vee b)$

$\equiv (\neg a \vee b)$

Stay on target:
We met our intermediate goal.
Don't forget the final goal!
We want to end up at $(\neg a \vee b)$

If we apply the distribution rule,
We'd get a $(\neg a \vee b)$

Commutative
Make the expression look exactly like the law (more on this later)

Negation
Simplifies the part we want to disappear.

Simplify $\mathrm{T} \wedge (\neg a \vee b)$ to $(\neg a \vee b)$

These identities hold for all propositions p, q, r

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Our First Proof

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b)$

$\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$

$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$

$\equiv (a \wedge b) \vee [\neg a]$

$\equiv [\neg a] \vee (a \wedge b)$

$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$

$\equiv (a \vee \neg a) \wedge (\neg a \vee b)$

$\equiv \mathrm{T} \wedge (\neg a \vee b)$

$\equiv (\neg a \vee b) \wedge \mathrm{T}$

$\equiv (\neg a \vee b)$

Stay on target:
We met our intermediate goal.
Don't forget the final goal!
We want to end up at $(\neg a \vee b)$

If we apply the distribution rule,
We'd get a $(\neg a \vee b)$

Commutative followed by **Identity**
Look exactly like the law, then apply it.

We're done!!!

Commutativity

We had the expression $(a \wedge b) \vee [\neg a]$

But before we applied the distributive law, we switched the order…why?

The law says $p \vee (q \wedge r) \equiv (\mathrm{p} \vee q) \wedge (p \vee r)$

not $(q \wedge r) \vee p \equiv (q \vee p) \wedge (r \vee p)$

So **technically** we needed to commute first.

Eventually (in about 2 weeks) we'll skip this step. For now, we're doing two separate steps.

Remember this is the "training wheel" stage. The point is to be careful.

More on Our First Proof

We now have an ironclad guarantee that

$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (\neg a \vee b)$

Hooray! But we could have just made a truth-table. Why a proof?

Here's one reason.

Proofs don't *just* give us an ironclad guarantee. They're also an explanation of *why* the claim is true.

The key insight to our simplification was "the last two pieces were the vacuous truth parts – the parts where p was false"

That's in there, *in the proof.*

Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b)$$

$\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$	Associative
$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$	Distributive
$\equiv (a \wedge b) \vee [\neg a \wedge \mathrm{T}]$	Negation
$\equiv (a \wedge b) \vee [\neg a]$	Identity
$\equiv [\neg a] \vee (a \wedge b)$	Commutative
$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$	Distributive
$\equiv (a \vee \neg a) \wedge (\neg a \vee b)$	Commutative
$\equiv \mathrm{T} \wedge (\neg a \vee b)$	Negation
$\equiv (\neg a \vee b) \wedge \mathrm{T}$	Commutative
$\equiv (\neg a \vee b)$	Identity

The last two terms are "vacuous truth" – they simplify to $\neg a$

a no longer matters in $a \wedge b$ if $\neg a$ automatically makes the expression true.

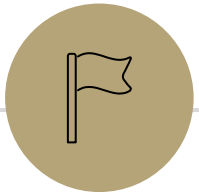
More on Our First Proof

With practice (and quite a bit of squinting) you can see not just the ironclad guarantee, but also the reason why something is true.

That's not easy with a truth table.

Proofs can also communicate intuition about why a statement is true.

We'll practice extracting intuition from proofs more this quarter.



Modifying Implications

Converse, Contrapositive

Implication:

If it's raining, then I have my umbrella.

$p \rightarrow q$

Converse:

If I have my umbrella, then it is raining.

$q \rightarrow p$

Contrapositive:

$\neg q \rightarrow \neg p$ If I don't have my umbrella, then it is not raining.

Inverse:

$\neg p \rightarrow \neg q$ If it is not raining, then I don't have my umbrella.

How do these relate to each other?

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T						
T	F						
F	T						
F	F						

Converse, Contrapositive

Implication:

$p \rightarrow q$

Converse:

$q \rightarrow p$

Contrapositive:

$\neg q \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg q$

An implication and its contrapositive have the same truth value!

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Contrapositive

We showed $p \rightarrow q \equiv \neg q \rightarrow \neg p$ with a truth table. Let's do a proof.

Try this one on your own. Remember

1. Know what you're trying to show.
2. Stay on target – take steps to get closer to your goal.

Hint: think about your tools.

There are lots of rules with AND/OR/NOT,

but very few with implications…

pollev.com/robbie

Help me adjust my explanation!

Contrapositive

$$
\begin{aligned}
p \rightarrow q &\equiv \neg p \vee q && \text{Law of Implication} \\
&\equiv q \vee \neg p && \text{Commutativity} \\
&\equiv \neg\neg q \vee \neg p && \text{Double Negation} \\
&\equiv \neg q \rightarrow \neg p && \text{Law of Implication}
\end{aligned}
$$

All of our rules deal with ORs and ANDs, let's switch the implication to just use AND/NOT/OR.

And do the same with our target

It's ok to work from both ends. In fact it's a very common strategy!

Now how do we get the top to look like the bottom?

Just a few more rules and we're done!

Work from both ends, but…

…make sure at the end, if you read from top-to-bottom, every step makes sense.

When proving an equivalence you must:
1. Start with the left side (or right side)

2. Modify what you had in the last step (using an equivalence)

3. Derive the right side (or left side if you started with the right)

You may **not** start with the equivalence you're trying to show, and simplify to something "obviously true."

More on why later in the quarter, but tl;dr for now is you can't use your goal as a starting assumption (it's what you're trying to show! If you knew it, no need to write a proof).