Section 09: Solutions

1. CFGs

Write a context-free grammar to match each of these languages.

(a) All binary strings that start with 11.
Solution:

$$\mathbf{S} \to 11\mathbf{T}$$
$$\mathbf{T} \to 1\mathbf{T} \mid 0\mathbf{T} \mid \varepsilon$$

(b) All binary strings that contain at most one 1.
Solution:

$$\mathbf{S} \to \mathbf{ABA}$$
$$\mathbf{A} \to 0\mathbf{A} \mid \varepsilon$$
$$\mathbf{B} \to 1 \mid \varepsilon$$

(c) All strings over 0, 1, 2 with the same number of 1s and 0s and exactly one 2.
Hint: Try modifying the grammar from Section 8 2c for binary strings with the same number of 1s and 0s (You may need to introduce new variables in the process).
Solution:

$$\mathbf{S} \to 2\mathbf{T} \mid \mathbf{T}2 \mid \mathbf{ST} \mid \mathbf{TS} \mid 0\mathbf{S}1 \mid 1\mathbf{S}0$$
$$\mathbf{T} \to \mathbf{TT} \mid 0\mathbf{T}1 \mid 1\mathbf{T}0 \mid \varepsilon$$

T is the grammar from Section 8 2c. It generates all binary strings with the same number of 1s and 0s. **S** matches a 2 at the beginning or end. The rest of the string must then match **T** since it cannot have another 2. If neither the first nor last character is a 2, then it falls into the usual cases of matching 0s and 1s, so we can mostly use the same rules as **T**. The main change is that **SS** becomes **ST** | **TS** to ensure that exactly one of the two parts contains a 2. The other change is that there is no ϵ since a 2 must appear somewhere.

2. Relations

(a) Consider the relation $R = \{(x, y) \ : \ x = y + 1\}$ on $\mathbb{N}$. Is R reflexive? Transitive? Symmetric? Anti-symmetric?
Solution:

It isn't reflexive, because $1 \neq 1 + 1$; so, $(1, 1) \notin R$. It isn't symmetric, because $(2, 1) \in R$ (because $2 = 1 + 1$), but $(1, 2) \notin R$, because $1 \neq 2 + 1$. It isn't transitive, because note that $(3, 2) \in R$ and $(2, 1) \in R$, but $(3,1) \notin R$. It is anti-symmetric because of the following: consider an arbitrary $(x,y) \in R$ where $x \neq y$. Then, $x = y+1$ by definition of R. However, $(y,x) \notin R$, because $y = x-1 \neq x+1$. Since $(x,y) \in R$ was arbitrary, R is anti-symmetric.

(b) Consider the relation $S = \{(x,y) \;:\; x^2 = y^2\}$ on $\mathbb{R}$. Prove that S is reflexive, transitive, and symmetric.
Solution:

Consider an arbitrary $x \in \mathbb{R}$. Note that by definition of equality, $x^2 = x^2$; so, $(x,x) \in S$; since $x \in R$ was arbitrary, S is reflexive.

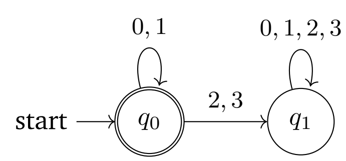
Consider an arbitrary $(x,y) \in S$. Then, $x^2 = y^2$. It follows that $y^2 = x^2$; so, $(y,x) \in S$. Since $(x,y) \in S$ was arbitrary, S is symmetric.

Consider an arbitrary $(x,y) \in S$ and an arbitrary $(y,z) \in S$. Then, $x^2 = y^2$, and $y^2 = z^2$. Since equality is transitive, $x^2 = z^2$. So, $(x,z) \in S$. Since $(x,y) \in S$ and $(y,z) \in S$ were arbitrary, S is transitive.

3. DFAs, Stage 1

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0,1,2,3\}$.

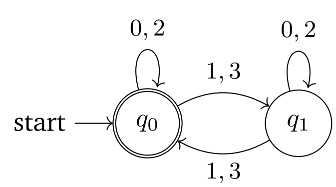
(a) All binary strings.
Solution:



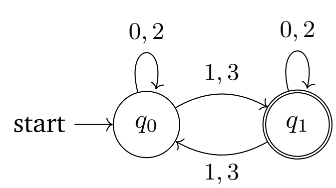
q_0: binary strings

q_1: strings that contain a character which is not 0 or 1.

(b) All strings whose digits sum to an even number.
Solution:



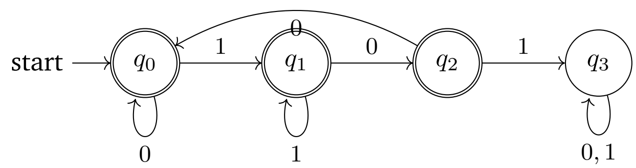
(c) All strings whose digits sum to an odd number.
Solution:



4. DFAs, Stage 2

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1\}$.

(a) All strings which do not contain the substring 101.
Solution:



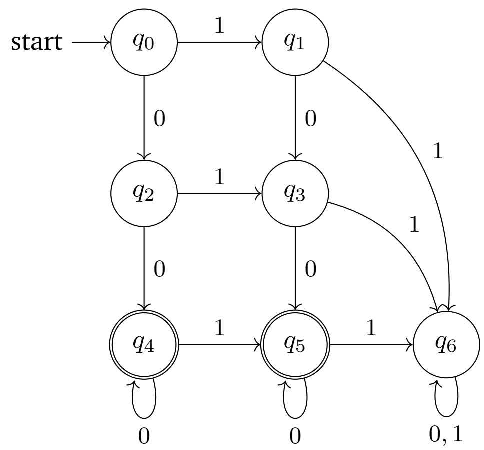
q_3: string that contain 101.

q_2: strings that don't contain 101 and end in 10.

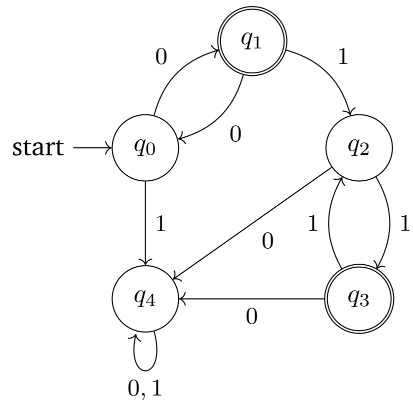
q_1: strings that don't contain 101 and end in 1.

q_0: ε, 0, strings that don't contain 101 and end in 00.

(b) All strings containing at least two 0's and at most one 1.
Solution:

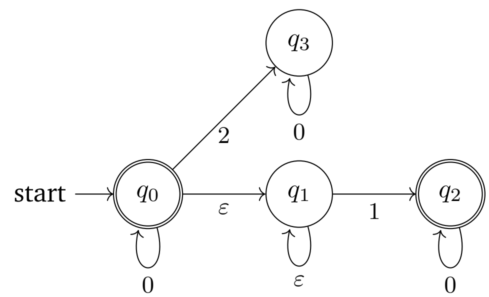


(c) All strings containing an even number of 1's and an odd number of 0's and not containing the substring 10.
Solution:



5. NFAs

(a) What language does the following NFA accept?



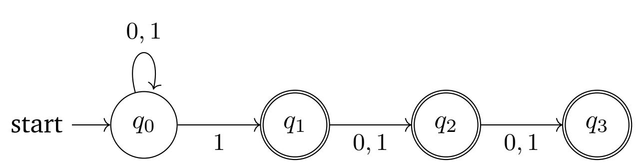
Solution:

All strings of only 0's and 1's not containing more than one 1.

(b) Create an NFA for the language "all binary strings that have a 1 as one of the last three digits".

Solution:

The following is one such NFA:

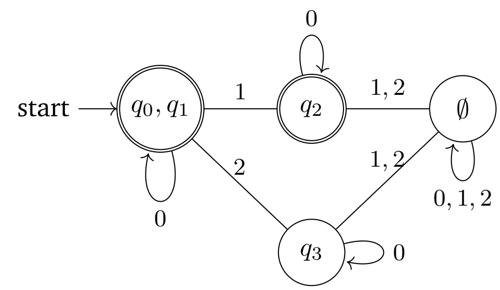


6. DFAs & Minimization

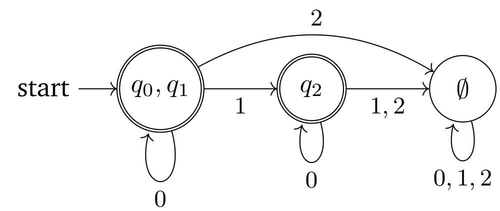
Note: We will not test you on minimization, although you may optionally read the extra slides and do this problem for fun

(a) Convert the NFA from 1a to a DFA, then minimize it.

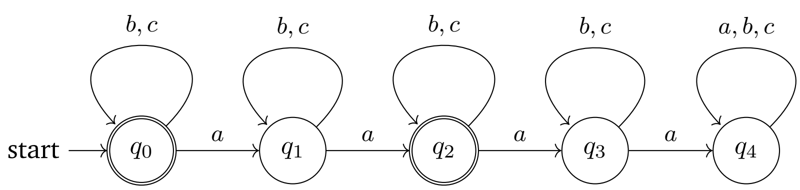
Solution:



Here is the minimized form:



(b) Minimize the following DFA:



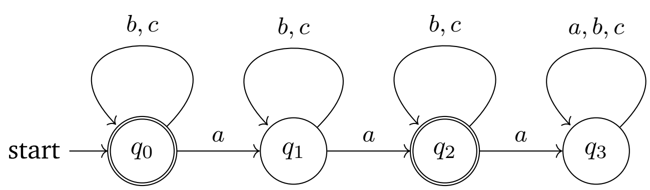
Solution:

Step 1: q_0, q_2 are final states and the rest are not final. So, we start with the initial partition with the following groups: group 1 is $\{q_0, q_2\}$ and group 2 is $\{q_1, q_3, q_4\}$.

Step 2: q_1 is sending a to group 1 while q_3, q_4 are sending a to group 2. So, we divide group 2. We get the following groups: group 1 is $\{q_0, q_2\}$, group 3 is $\{q_1\}$ and group 4 is $\{q_3, q_4\}$.

Step 3: q_0 is sending a to group 3 and q_2 is sending a to group 4. So, we divide group 1. We will have the following groups: group 3 is $\{q_1\}$, group 4 is $\{q_3, q_4\}$, group 5 is $\{q_0\}$ and group 6 is $\{q_2\}$.

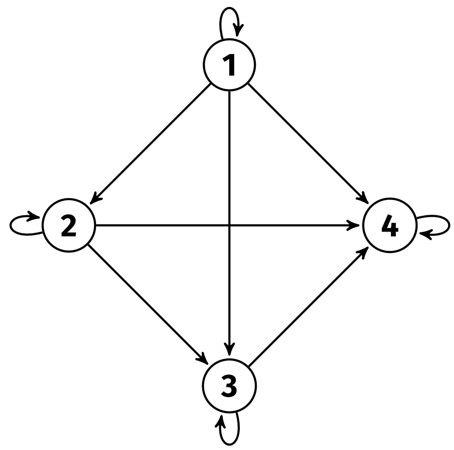
The minimized DFA is the following:



7. More Relations

Note: We will not test you nor give you homework problems based on the following types of relation problems, however, you may still attempt these problems for fun, using the lecture slides.

(a) Draw the transitive-reflexive closure of $\{(1, 2), (2, 3), (3, 4)\}$.
Solution:



(b) Suppose that R is reflexive. Prove that $R \subseteq R^2$.
Solution:

Suppose $(a, b) \in R$. Since R is reflexive, we know $(b, b) \in R$ as well. Since there is a b such that $(a, b) \in R$ and $(b, b) \in R$, it follows that $(a, b) \in R^2$. Thus, $R \subseteq R^2$.