Section 07: Induction

0. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1 \\ f(n) &= 2f(n-1) - f(n-2) \text{ for } n \geq 2 \end{aligned}$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

1. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

2. Induction with Formulas

These problems are a little more difficult and abstract. Try making sure you can do all the other problems before trying these ones.

(a) (i) Show that given two sets A and B that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don't use induction.)

(ii) Show using induction that for an integer $n \geq 2$, given n sets $A_1, A_2, \ldots, A_{n-1}, A_n$ that

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}$$

(b) (i) Show that given any integers a, b, and c, if $c \mid a$ and $c \mid b$, then $c \mid (a + b)$. (Don't use induction.)

(ii) Show using induction that for any integer $n \geq 2$, given n numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \ldots, n$, that

$$c \mid (a_1 + a_2 + \cdots + a_{n-1} + a_n).$$

In other words, if a number divides each term in a sum then that number divides the sum.

3. Structural Induction

(a) Consider the following recursive definition of strings.

Basis Step: "" is a string

Recursive Step: If X is a string and c is a character then $\mathsf{append}(c, X)$ is a string.

Recall the following recursive definition of the function len:

$$\begin{aligned} \mathsf{len}(\text{""}) &= 0 \\ \mathsf{len}(\mathsf{append}(c, X)) &= 1 + \mathsf{len}(X) \end{aligned}$$

Now, consider the following recursive definition:

$$\begin{aligned}
\mathsf{double}(\texttt{""}) &= \texttt{""} \\
\mathsf{double}(\mathsf{append}(c, X)) &= \mathsf{append}(c, \mathsf{append}(c, \mathsf{double}(X))).
\end{aligned}$$

Prove that for any string X, $\mathsf{len}(\mathsf{double}(X)) = 2\mathsf{len}(X)$.

(b) Consider the following definition of a (binary) **Tree**:

Basis Step: $\bullet$ is a **Tree**.

Recursive Step: If L is a **Tree** and R is a **Tree** then $\texttt{Tree}(\bullet, L, R)$ is a **Tree**.

The function leaves returns the number of leaves of a **Tree**. It is defined as follows:

$$\begin{aligned}
\mathsf{leaves}(\bullet) &= 1 \\
\mathsf{leaves}(\texttt{Tree}(\bullet, L, R)) &= \mathsf{leaves}(L) + \mathsf{leaves}(R)
\end{aligned}$$

Also, recall the definition of size on trees:

$$\begin{aligned}
\mathsf{size}(\bullet) &= 1 \\
\mathsf{size}(\texttt{Tree}(\bullet, L, R)) &= 1 + \mathsf{size}(L) + \mathsf{size}(R)
\end{aligned}$$

Prove that $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$ for all Trees T.

(c) Prove the previous claim using strong induction. Define $P(n)$ as "all trees T of size n satisfy $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$". You may use the following facts:

- For any tree T we have $\mathsf{size}(T) \geq 1$.
- For any tree T, $\mathsf{size}(T) = 1$ if and only if $T = \bullet$.

If we wanted to prove these claims, we could do so by structural induction.

Note, in the inductive step you should start by letting T be an arbitrary tree of size $k + 1$.