Section 07: Solutions

0. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$
\begin{aligned}
f(0) &= 0 \\
f(1) &= 1 \\
f(n) &= 2f(n-1) - f(n-2) \text{ for } n \geq 2
\end{aligned}
$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

Solution:

Let $P(n)$ be "$f(n) = n$". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on n.

Base Cases $(n = 0, n = 1)$**:** $f(0) = 0$ and $f(1) = 1$ by definition.

Inductive Hypothesis: Assume that $P(0) \wedge P(1) \wedge \ldots P(k)$ hold for some arbitrary $k \geq 1$.

Inductive Step: We show $P(k+1)$:

$$
\begin{aligned}
f(k+1) &= 2f(k) - f(k-1) && \text{[Definition of } f\text{]} \\
&= 2(k) - (k-1) && \text{[Induction Hypothesis]} \\
&= k+1 && \text{[Algebra]}
\end{aligned}
$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by principle of strong induction.

1. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$. **Solution:**

Let $P(n)$ be "$6n + 6 < 2^n$". We will prove $P(n)$ for all integers $n \geq 6$ by induction on n

Base Case $(n = 6)$**:** $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Inductive Hypothesis: Assume that $6k + 6 < 2^k$ for an arbitrary integer $k \geq 6$.

Inductive Step: $\boxed{\text{Goal: Show } 6(k+1) + 6 < 2^{k+1}}$

$$
\begin{aligned}
6(k+1) + 6 &= 6k + 6 + 6 \\
&< 2^k + 6 && \text{[Inductive Hypothesis]} \\
&< 2^k + 2^k && \text{[Since } 2^k > 6 \text{, since } k \geq 6\text{]} \\
&= 2 \cdot 2^k \\
&= 2^{k+1}
\end{aligned}
$$

So $P(k) \rightarrow P(k+1)$ for an arbitrary integer $k \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by the principle of induction.

2. Induction with Formulas

These problems are a little more difficult and abstract. Try making sure you can do all the other problems before trying these ones.

(a) (i) Show that given two sets A and B that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don't use induction.)

Solution:

Let x be arbitrary. Then,

$$\begin{aligned} x \in \overline{A \cup B} &\equiv \neg(x \in A \cup B) && \text{[Definition of complement]} \\ &\equiv \neg(x \in A \vee x \in B) && \text{[Definition of union]} \\ &\equiv \neg(x \in A) \wedge \neg(x \in B) && \text{[De Morgan's Laws]} \\ &\equiv x \in \overline{A} \wedge x \in \overline{B} && \text{[Definition of complement]} \\ &\equiv x \in (\overline{A} \cap \overline{B}) && \text{[Definition of intersection]} \end{aligned}$$

Since x was arbitrary we have that $x \in \overline{A \cup B}$ if and only if $x \in \overline{A} \cap \overline{B}$ for all x. By the definition of set equality we've shown,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

(ii) Show using induction that for an integer $n \geq 2$, given n sets $A_1, A_2, \ldots, A_{n-1}, A_n$ that

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}$$

Solution:

Let $P(n)$ be "given n sets $A_1, A_2, \ldots, A_{n-1}, A_n$ it holds that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}} \cap \overline{A_n}$." We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $P(2)$ says that for two sets A_1 and A_2 that $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$, which is exactly part (a) so $P(2)$ holds.

Inductive Hypothesis: Suppose that $P(k)$ holds for some arbitrary integer $k \geq 2$.

Inductive Step: Let $A_1, A_2, \ldots, A_k, A_{k+1}$ be sets. Then by part (a) we have,

$$\overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} = \overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}}.$$

By the inductive hypothesis we have $\overline{A_1 \cup A_2 \cup \cdots A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}$. Thus,

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}} = (\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_k}) \cap \overline{A_{k+1}}.$$

We've now shown

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}} = \overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_k} \cap \overline{A_{k+1}}.$$

which is exactly $P(k+1)$.

Conclusion $P(n)$ holds for all integers $n \geq 2$ by the principle of induction.

(b) (i) Show that given any integers a, b, and c, if $c \mid a$ and $c \mid b$, then $c \mid (a+b)$. (Don't use induction.)

Solution:

Let a, b, and c be arbitrary integers and suppose that $c \mid a$ and $c \mid b$. Then by definition there exist integers j and k such that $a = jc$ and $b = kc$. Then $a + b = jc + kc = (j + k)c$. Since $j + k$ is an integer, by definition we have $c \mid (a + b)$.

(ii) Show using induction that for any integer $n \geq 2$, given n numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \ldots, n$, that

$$c \mid (a_1 + a_2 + \cdots + a_{n-1} + a_n).$$

In other words, if a number divides each term in a sum then that number divides the sum.

Solution:

Let $P(n)$ be "given n numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \ldots, n$, it holds that $c \mid (a_1 + a_2 + \cdots + a_n)$." We show $P(n)$ holds for all integer $n \geq 2$ by induction on n.

Base Case: $P(2)$ says that given two integers a_1 and a_2, for any integer c such that $c \mid a_1$ and $c \mid a_2$ it holds that $c \mid (a_1 + a_2)$. This is exactly part (a) so $P(2)$ holds.

Inductive Hypothesis: Suppose that $P(k)$ holds for some arbitrary integer $k \geq 2$.

Inductive Step: Let $a_1, a_2, \ldots, a_k, a_{k+1}$ be $k + 1$ integers. Let c be arbitrary and suppose that $c \mid a_i$ for $i = 1, 2, \ldots, k + 1$. Then we can write

$$a_1 + a_2 + \cdots + a_k + a_{k+1} = (a_1 + a_2 + \cdots + a_k) + a_{k+1}.$$

The sum $a_1 + a_2 + \cdots + a_k$ has k terms and c divides all of them, meaning we can apply the inductive hypothesis. It says that $c \mid (a_1 + a_2 + \cdots + a_k)$. Since $c \mid (a_1 + a_2 + \cdots + a_k)$ and $c \mid a_{k+1}$, by part (a) we have,

$$c \mid (a_1 + a_2 + \cdots + a_k + a_{k+1}).$$

This shows $P(k + 1)$.

Conclusion: $P(n)$ holds for all integers $n \geq 2$ by induction the principle of induction.

3. Structural Induction

(a) Consider the following recursive definition of strings.

Basis Step: "" is a string

Recursive Step: If X is a string and c is a character then $\mathsf{append}(c, X)$ is a string.

Recall the following recursive definition of the function len:

$$\begin{aligned} \mathsf{len}(\texttt{""}) &= 0 \\ \mathsf{len}(\mathsf{append}(c, X)) &= 1 + \mathsf{len}(X) \end{aligned}$$

Now, consider the following recursive definition:

$$\begin{aligned} \mathsf{double}(\texttt{""}) &= \texttt{""} \\ \mathsf{double}(\mathsf{append}(c, X)) &= \mathsf{append}(c, \mathsf{append}(c, \mathsf{double}(X))). \end{aligned}$$

Prove that for any string X, $\mathsf{len}(\mathsf{double}(X)) = 2\mathsf{len}(X)$.

Solution:

For a string X, let $\mathsf{P}(X)$ be "$\mathsf{len}(\mathsf{double}(X)) = 2\mathsf{len}(X)$". We prove $\mathsf{P}(X)$ for all strings X by structural induction on X.

Base Case (X = ""): By definition, $\mathsf{len}(\mathsf{double}(\texttt{""})) = \mathsf{len}(\texttt{""}) = 0 = 2 \cdot 0 = 2\mathsf{len}(\texttt{""})$, so $\mathsf{P}(\texttt{""})$ holds

Inductive Hypothesis: Suppose $\mathsf{P}(X)$ holds for some arbitrary string X.

Inductive Step: $\boxed{\text{Goal: Show that } \mathsf{P}(\mathsf{append}(c, X)) \text{ holds for any character } c.}$

$$
\begin{aligned}
\mathsf{len}(\mathsf{double}(\mathsf{append}(c, X))) &= \mathsf{len}(\mathsf{append}(c, \mathsf{append}(c, \mathsf{double}(X)))) && \text{[By Definition of } \mathsf{double}] \\
&= 1 + \mathsf{len}(\mathsf{append}(c, \mathsf{double}(X))) && \text{[By Definition of } \mathsf{len}] \\
&= 1 + 1 + \mathsf{len}(\mathsf{double}(X)) && \text{[By Definition of } \mathsf{len}] \\
&= 2 + 2\mathsf{len}(X) && \text{[By IH]} \\
&= 2(1 + \mathsf{len}(X)) && \text{[Algebra]} \\
&= 2(\mathsf{len}(\mathsf{append}(c, X))) && \text{[By Definition of } \mathsf{len}]
\end{aligned}
$$

This proves $\mathsf{P}(\mathsf{append}(c, X))$.

Conclusion: $\mathsf{P}(X)$ holds for all strings X by structural induction.

(b) Consider the following definition of a (binary) **Tree**:

Basis Step: $\bullet$ is a **Tree**.

Recursive Step: If L is a **Tree** and R is a **Tree** then $\texttt{Tree}(\bullet, L, R)$ is a **Tree**.

The function leaves returns the number of leaves of a **Tree**. It is defined as follows:

$$
\begin{aligned}
\mathsf{leaves}(\bullet) &= 1 \\
\mathsf{leaves}(\texttt{Tree}(\bullet, L, R)) &= \mathsf{leaves}(L) + \mathsf{leaves}(R)
\end{aligned}
$$

Also, recall the definition of size on trees:

$$
\begin{aligned}
\mathsf{size}(\bullet) &= 1 \\
\mathsf{size}(\texttt{Tree}(\bullet, L, R)) &= 1 + \mathsf{size}(L) + \mathsf{size}(R)
\end{aligned}
$$

Prove that $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$ for all Trees T.

Solution:

For a tree T, let $\mathsf{P}(T)$ be $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$. We prove $\mathsf{P}(T)$ for all trees T by structural induction on T.

Base Case (T = $\bullet$): By definition of $\mathsf{leaves}(\bullet)$, $\mathsf{leaves}(\bullet) = 1$ and $\mathsf{size}(\bullet) = 1$. So, $\mathsf{leaves}(\bullet) = 1 \geq 1/2 + 1/2 = \mathsf{size}(\bullet)/2 + 1/2$, so $\mathsf{P}(\bullet)$ holds.

Inductive Hypothesis: Suppose $\mathsf{P}(L)$ and $\mathsf{P}(R)$ hold for some arbitrary trees L, R.

Inductive Step: $\boxed{\text{Goal: Show that } \mathsf{P}(\texttt{Tree}(\bullet, L, R)) \text{ holds.}}$

$$
\begin{aligned}
\mathsf{leaves}(\texttt{Tree}(\bullet, L, R)) &= \mathsf{leaves}(L) + \mathsf{leaves}(R) && \text{[By Definition of } \mathsf{leaves}] \\
&\geq (\mathsf{size}(L)/2 + 1/2) + (\mathsf{size}(R)/2 + 1/2) && \text{[By IH]} \\
&= (1/2 + \mathsf{size}(L)/2 + \mathsf{size}(R)/2) + 1/2 && \text{[By Algebra]} \\
&= \frac{1 + \mathsf{size}(L) + \mathsf{size}(R)}{2} + 1/2 && \text{[By Algebra]} \\
&= \mathsf{size}(T)/2 + 1/2 && \text{[By Definition of } \mathsf{size}]
\end{aligned}
$$

This proves $\mathsf{P}(\texttt{Tree}(\bullet, L, R))$.

Conclusion: Thus, $\mathsf{P}(T)$ holds for all trees T by structural induction.

(c) Prove the previous claim using strong induction. Define $P(n)$ as "all trees T of size n satisfy $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$". You may use the following facts:

- For any tree T we have $\mathsf{size}(T) \geq 1$.
- For any tree T, $\mathsf{size}(T) = 1$ if and only if $T = \bullet$.

If we wanted to prove these claims, we could do so by structural induction.

Note, in the inductive step you should start by letting T be an arbitrary tree of size $k+1$.

Solution:

Let $P(n)$ be "all trees T of size n satisfy $\mathsf{leaves}(T) \geq \mathsf{size}(T)/2 + 1/2$". We show $P(n)$ for all integers $n \geq 1$ by strong induction on n.

Base Case: Let T be an arbitrary tree of size 1. The only tree with size 1 is $\bullet$, so $T = \bullet$. By definition, $\mathsf{leaves}(T) = \mathsf{leaves}(\bullet) = 1$ and thus $\mathsf{size}(T) = 1 = 1/2 + 1/2 = \mathsf{size}(T)/2 + 1/2$. This shows the base case holds.

Inductive Hypothesis: Suppose that $P(j)$ holds for all integers $j = 1, 2, \ldots, k$ for some arbitrary integer $k \geq 1$.

Inductive Step: Let T be an arbitrary tree of size $k+1$. Since $k+1 > 1$, we must have $T \neq \bullet$. It follows from the definition of a tree that $T = \texttt{Tree}(\bullet, L, R)$ for some trees L and R. By definition, we have $\mathsf{size}(T) = 1 + \mathsf{size}(L) + \mathsf{size}(R)$. Since sizes are non-negative, this equation shows $\mathsf{size}(T) > \mathsf{size}(L)$ and $\mathsf{size}(T) > \mathsf{size}(R)$ meaning we can apply the inductive hypothesis. This says that $\mathsf{leaves}(L) \geq \mathsf{size}(L)/2 + 1/2$ and $\mathsf{leaves}(R) \geq \mathsf{size}(R)/2 + 1/2$.

We have,

$$
\begin{aligned}
\mathsf{leaves}(T) &= \mathsf{leaves}(\texttt{Tree}(\bullet, L, R)) \\
&= \mathsf{leaves}(L) + \mathsf{leaves}(R) && \text{[By Definition of } \mathsf{leaves}] \\
&\geq (\mathsf{size}(L)/2 + 1/2) + (\mathsf{size}(R)/2 + 1/2) && \text{[By IH]} \\
&= (1/2 + \mathsf{size}(L)/2 + \mathsf{size}(R)/2) + 1/2 && \text{[By Algebra]} \\
&= \frac{1 + \mathsf{size}(L) + \mathsf{size}(R)}{2} + 1/2 && \text{[By Algebra]} \\
&= \mathsf{size}(T)/2 + 1/2 && \text{[By Definition of } \mathsf{size}]
\end{aligned}
$$

This shows $P(k+1)$.

Conclusion: $P(n)$ holds for all integers $n \geq 1$ by the principle of strong induction.

Note, this proves the claim for all trees because every tree T has some size $s \geq 1$. Then $P(s)$ says that all trees of size s satisfy the claim, including T.