CSE 311 Section 5

Number Theory & Induction

Administrivia & Introductions

Announcements & Reminders

- HW3
  - If you think something was graded incorrectly, submit a regrade request!
- HW4 due yesterday 10PM on Gradescope
  - Use late days if you need them!
- HW5
  - 2 parts!
  - BOTH PARTS due Wednesday 2/8 @ 10pm
- Midterm is Next Weekend! (Friday 2/10 – Sunday 2/12)
  - "Take Home" exam on Gradescope
  - You will have 90 minutes to complete it, starting from when you open it on Gradescope
  - It is designed to take ~30 minutes

References

- Helpful reference sheets can be found on the course website!
  - https://courses.cs.washington.edu/courses/cse311/23wi/resources/
- How to LaTeX (found on Assignments page of website):
  - https://courses.cs.washington.edu/courses/cse311/23wi/assignments/HowToLaTeX.pdf
- Set Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/23wi/resources/reference-sets.pdf
- Number Theory Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/23wi/resources/reference-number-theory.pdf
- Induction Templates
  - https://courses.cs.washington.edu/courses/cse311/22sp/resources/induction-templates.pdf
- Plus more!

Number Theory

Some Definitions

- Divides:
  - For $a, b \in \mathbb{Z}$: $a \mid b$ iff $\exists(k \in \mathbb{Z})\ b = ka$
  - For integers a and b, we say a divides b if and only if there exists an integer k such that $b = ka$

- Congruence Modulo:
  - For $a, b \in \mathbb{Z}, m \in \mathbb{Z}^+$: $a \equiv b \pmod{m}$ iff $m \mid (b - a)$
  - For integers a and b and positive integer m, we say a is congruent to b modulo m if and only if m divides $b - a$

Problem 5 – Modular Arithmetic

a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Lets walk through part (a) together.

Problem 5 – Modular Arithmetic

a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

Suppose that $a \mid b$ and $b \mid a$, where a, b are integers.

$\dots$

Therefore, it follows that $a = -b$ or $a = b$.

Start with your proof skeleton!

Problem 5 – Modular Arithmetic

a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

Suppose that $a \mid b$ and $b \mid a$, where a, b are integers.

By the definition of divides, we have $a \neq 0, b \neq 0$ and $b = ka, a = jb$ for some integers k, j.
$\ldots$

Therefore, it follows that $a = -b$ or $a = b$.

Problem 5 – Modular Arithmetic

a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

Suppose that $a \mid b$ and $b \mid a$, where a, b are integers.

By the definition of divides, we have $a \neq 0, b \neq 0$ and $b = ka, a = jb$ for some integers k, j.
Combining these equations, we see that $a = j(ka)$.
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Therefore, it follows that $a = -b$ or $a = b$.

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Then, dividing both sides by a, we get $1 = jk$. So, $\frac{1}{j} = k$.
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Suppose that $a \mid b$ and $b \mid a$, where a, b are integers.

By the definition of divides, we have $a \neq 0, b \neq 0$ and $b = ka, a = jb$ for some integers k, j.
Combining these equations, we see that $a = j(ka)$.
Then, dividing both sides by a, we get $1 = jk$. So, $\frac{1}{j} = k$.
Note that j and k are integers, which is only possible if $j, k \in \{1, -1\}$.

Therefore, it follows that $a = -b$ or $a = b$.

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a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Now try part (b) with the people around you, and then we'll go over it together!

Problem 5 – Modular Arithmetic

b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

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b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$.

…

Therefore, we have $a \equiv b \pmod{n}$.

Problem 5 – Modular Arithmetic

b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$.

…

… we have $n \mid (b - a)$.

Therefore, by definition of congruence, we have $a \equiv b \pmod{n}$.

Problem 5 – Modular Arithmetic

b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$.

NOTE: we don't know what C will look like yet, just that there is SOME integer here!

…

… we have $b - a = nC$.

By definition of divides, we have $n \mid (b - a)$.

Therefore, by definition of congruence, we have $a \equiv b \pmod{n}$.

Problem 5 – Modular Arithmetic

b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$.

By definition of divides, we have $m = kn$ for some $k \in \mathbb{Z}$.

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Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$.

By definition of divides, we have $m = kn$ for some $k \in \mathbb{Z}$.
By definition of congruence, we have $m \mid a - b$, which means that $a - b = mj$ for some $j \in \mathbb{Z}$.
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… we have $b - a = nC$.
By definition of divides, we have $n \mid (b - a)$.

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By definition of congruence, we have $m \mid a - b$, which means that $a - b = mj$ for some $j \in \mathbb{Z}$.
Combining the two equations, we see that $a - b = (knj) = n(kj)$.
… we have $b - a = nC$.
By definition of divides, we have $n \mid (b - a)$.

Therefore, by definition of congruence, we have $a \equiv\ b \pmod{n}$.

Problem 5 – Modular Arithmetic

b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$.

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By definition of congruence, we have $m \mid a - b$, which means that $a - b = mj$ for some $j \in \mathbb{Z}$.
Combining the two equations, we see that $a - b = (knj) = n(kj)$.
Equivalently, we have $b - a = n(-kj)$.
By definition of divides, we have $n \mid (b - a)$.

Therefore, by definition of congruence, we have $a \equiv \ b \pmod{n}$.

Problem 5 – Modular Arithmetic

b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod{m}$, where a and b are integers, then $a \equiv b \pmod{n}$.

Suppose $n \mid m$ with $n, m > 1$, and $a \equiv b \pmod{m}$.

By definition of divides, we have $m = kn$ for some $k \in \mathbb{Z}$.
By definition of congruence, we have $m \mid a - b$, which means that $a - b = mj$ for some $j \in \mathbb{Z}$.
Combining the two equations, we see that $a - b = (knj) = n(kj)$.
Equivalently, we have $b - a = n(-kj)$.
Because $-kj$ is an integer, by definition of divides, we have $n \mid (b - a)$.

Therefore, by definition of congruence, we have $a \equiv\ b \pmod{n}$.

Induction

Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all n by induction on n.

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds for all n by the principle of induction.

Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds **for all n** by induction on n.

Note: often you will condition n here, like "all natural numbers n" or "$n \geq 0$"

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds **for all n** by the principle of induction.

Match the earlier condition on n in your conclusion!

Problem 6 – Induction with Equality

a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Lets walk through part (a) together.

We can “fill in” our induxtion template to construct our proof by induction.

Problem 6 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "". We show $P(n)$ holds for (some) n by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

Problem 6 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

<u>Base Case:</u> $P(b)$:

<u>Inductive Hypothesis:</u> Suppose $P(k)$ holds for an arbitrary $k \geq b$.

<u>Inductive Step:</u> Goal: Show $P(k + 1)$:

<u>Conclusion:</u> Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

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Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 6 – Induction with Equality

Show using induction that $0+1+2+\cdots+n=\frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

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<u>Inductive Step:</u> Goal: Show $P(k+1)$: $0+1+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}$

$$0+1+\cdots+k+(k+1)=\cdots$$

$$\cdots$$

$$=\frac{(k+1)(k+2)}{2}$$

?

<u>Conclusion:</u> Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

$$0 + 1 + \cdots + k + (k+1) = (0 + 1 + \cdots + k) + (k+1)$$

$$\ldots$$

$$= \frac{(k+1)(k+2)}{2}$$

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Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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$$
\begin{aligned}
0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\
&\dots \\
&= \frac{(k+1)(k+2)}{2} && ?
\end{aligned}
$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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&= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&\dots \\
&= \frac{(k+1)(k+2)}{2} && ?
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&= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} && ?
\end{aligned}
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Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} && \text{factoring out (k+1)}
\end{aligned}
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Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Now try part (b) with people around you, and then we'll go over it together!

Problem 6 – Induction with Equality

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

$\triangle_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:

$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Let $P(n)$ be "". We show $P(n)$ holds for (some) n by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

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$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

$\triangle_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:

$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(b)$:

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Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0^3 = 0 = (0)^2$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

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Problem 6 – Induction with Equality

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

$\triangle_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:

$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0^3 = 0 = (0)^2$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step: Goal: Show $P(k+1)$: $0^3 + 1^3 + \cdots + k^3 + (k+1)^3 = (0 + 1 + \cdots + k + (k+1))^2$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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$$
\begin{aligned}
0^3 + 1^3 + \cdots + k^3 + (k+1)^3 &= (0 + 1 + \cdots + k + (k+1))^2 + (k+1)^3 && \text{by I.H.} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{by (a)} \\
&= (k+1)^2\left(\frac{k^2}{2^2} + (k+1)\right) && \text{factor out } (k+1)^2 \\
&= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) && \\
&= (k+1)^2\left(\frac{(k+2)^2}{4}\right) && \text{factor numerator} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \\
&= (0 + 1 + \cdots + k + (k+1))^2 && \text{by (a)}
\end{aligned}
$$

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That’s All, Folks!

Thanks for coming to section this week!
Any questions?