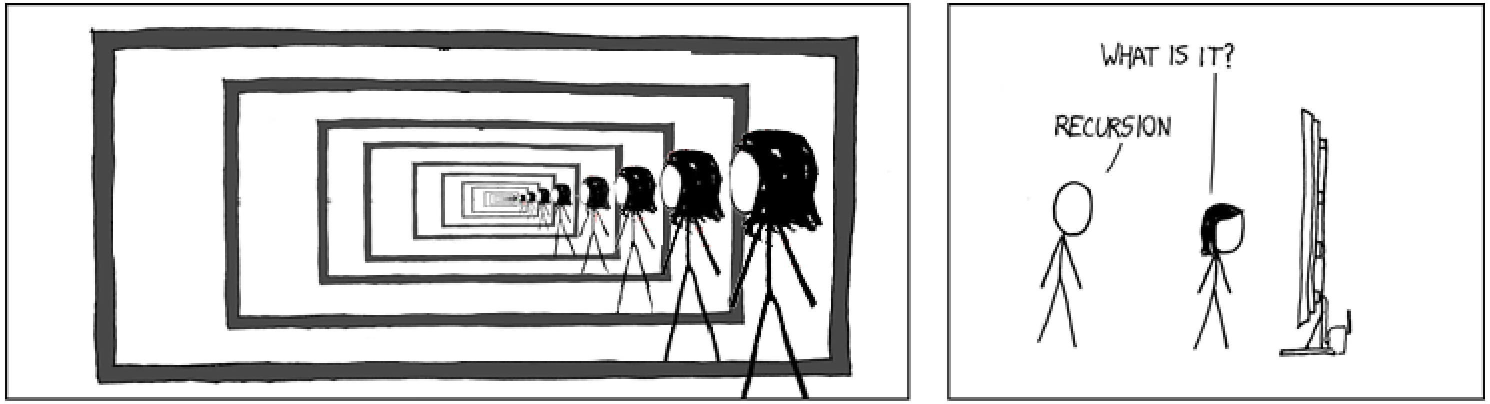

Structural Induction

CSE 311 Winter 23
Lecture 17

Induction Big Picture

So far: We used induction to prove a statement over the natural numbers.

“Prove that P(n) holds for all natural numbers n.”

Next goal: In CS, we deal with Strings, Lists, Trees, and other recursively defined sets. Would like to prove statements over these sets.

“Prove that P(T) holds for all trees T.”

“Prove that P(x) holds for all strings x.”

Recursive Definitions of Sets

Define a set S as follows:

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 2 \in S$.

Exclusion Rule: Every element of S follows from the basis step or a finite number of recursive steps.

Q1: What is S?

Q2: Why do we need the exclusion rule?

Recursive Definitions of Sets

All Natural Numbers

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 1 \in S$.

All Integers

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$.

Integer coordinates in the line $y = x$

Basis Step: $(0,0) \in S$

Recursive Step: If $(x, y) \in S$ then $(x + 1, y + 1) \in S$ and $(x - 1, y - 1) \in S$.

Recursive Definitions of Sets

Q1: What is this set?

Basis Step: $6 \in S, 15 \in S$

Recursive Step: If $x, y \in S$ then $x + y \in S$

Q2: Write a recursive definition for the set of powers of 3 $\{1,3,9,27, \dots\}$

Basis Step:

Recursive Step:

Structural Induction

Goal is to prove $P(s)$ for all $s \in S$…

Base Case: Show $P(b)$ for all elements b in the basis step.

Inductive Hypothesis: Assume $P()$ holds for arbitrary element(s) that we've already constructed

Inductive Step: Prove that $P()$ holds for a new element constructed using the recursive step

Structural Induction Example

Let S be:
Basis: $6 \in \mathrm{S}, 15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$.

Show by structural induction that every element of S is divisible by 3.

Structural Induction

Basis: $6 \in S, 15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$.

1. Intro:
2. Base Case(s):

3. Inductive Hypothesis:

4. Inductive Step:

5. Conclusion:

Structural Induction

Basis: $6 \in \mathrm{S}, 15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$.

1. Intro: Let $P(x)$ be x is divisible by 3. We show $P(x)$ holds for all $x \in S$ by structural induction.

2. Base Case(s): $6 = 2 \cdot 3$ so $3|6$, and $P(6)$ holds. $15 = 5 \cdot 3$, so $3|15$ and $P(15)$ holds.

3. Inductive Hypothesis:

4. Inductive Step:

5. Conclusion: We conclude $P(x) \forall x \in \mathrm{S}$ by the principle of induction.

Structural Induction

Basis: $6 \in S, 15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$.

1. Intro: Let $P(x)$ be x is divisible by 3. We show $P(x)$ holds for all $x \in S$ by structural induction.

2. Base Case(s): $6 = 2 \cdot 3$ so $3|6$, and $P(6)$ holds. $15 = 5 \cdot 3$, so $3|15$ and $P(15)$ holds.

3. Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ for arbitrary $x, y \in S$.

4. Inductive Step:

5. Conclusion: We conclude $P(x) \forall x \in S$ by the principle of induction.

Structural Induction

Basis: $6 \in S, 15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$.

1. Intro: Let $P(x)$ be x is divisible by 3. We show $P(x)$ holds for all $x \in S$ by structural induction.

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3. Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ for arbitrary $x, y \in S$.

4. Inductive Step: Goal $P(x + y)$ holds

By IH $3|\mathrm{x}$ and $3|\mathrm{y}$. So $x = 3n$ and $y = 3m$ for integers m, n.

Adding the equations, $x + y = 3(n + m)$. Since n, m are integers, we have $3|(x + y)$ by definition of divides. This gives $P(x + y)$.

5. Conclusion: We conclude $P(x) \forall x \in \mathrm{S}$ by the principle of induction.

Structural Induction Template

1. Define $P()$ Show that $P(x)$ holds for all $x \in S$. State your proof is by structural induction.

2. Base Case: Show $P(x)$ for all base cases x in S.

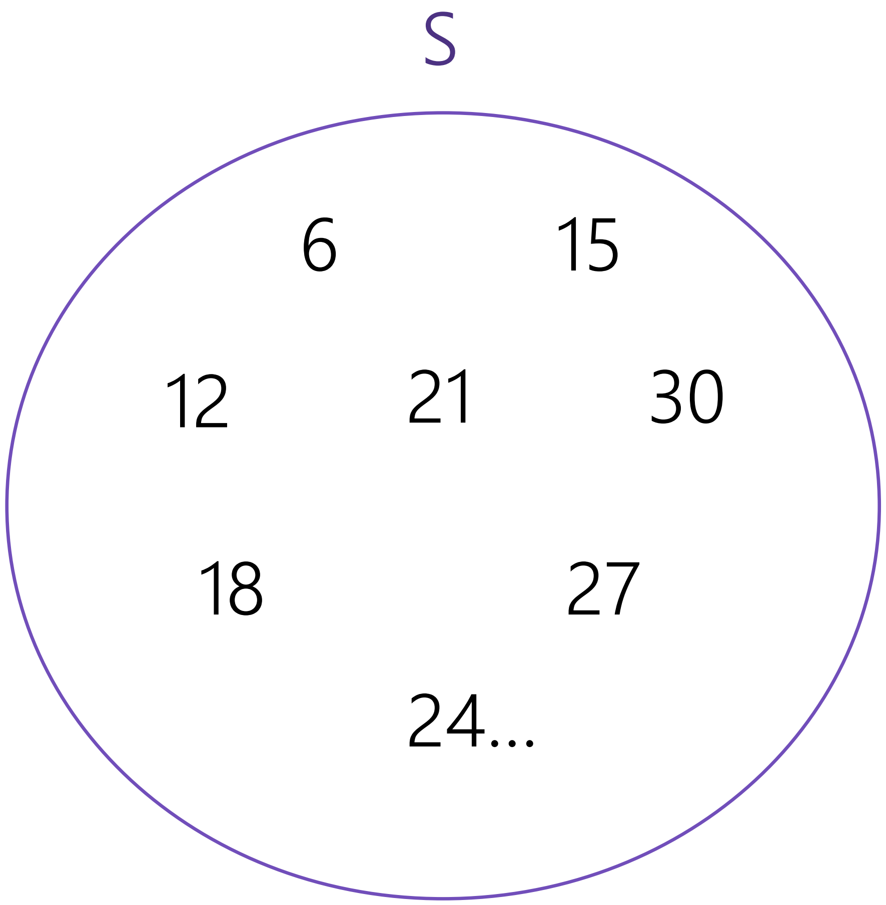
3. Inductive Hypothesis: Suppose $P(x)$ for all x listed as in S in the recursive rules.

4. Inductive Step: Show $P()$ holds for the “new element” given.

You will need a separate step for every rule.

5. Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.

Wait a minute! Why can we do this?



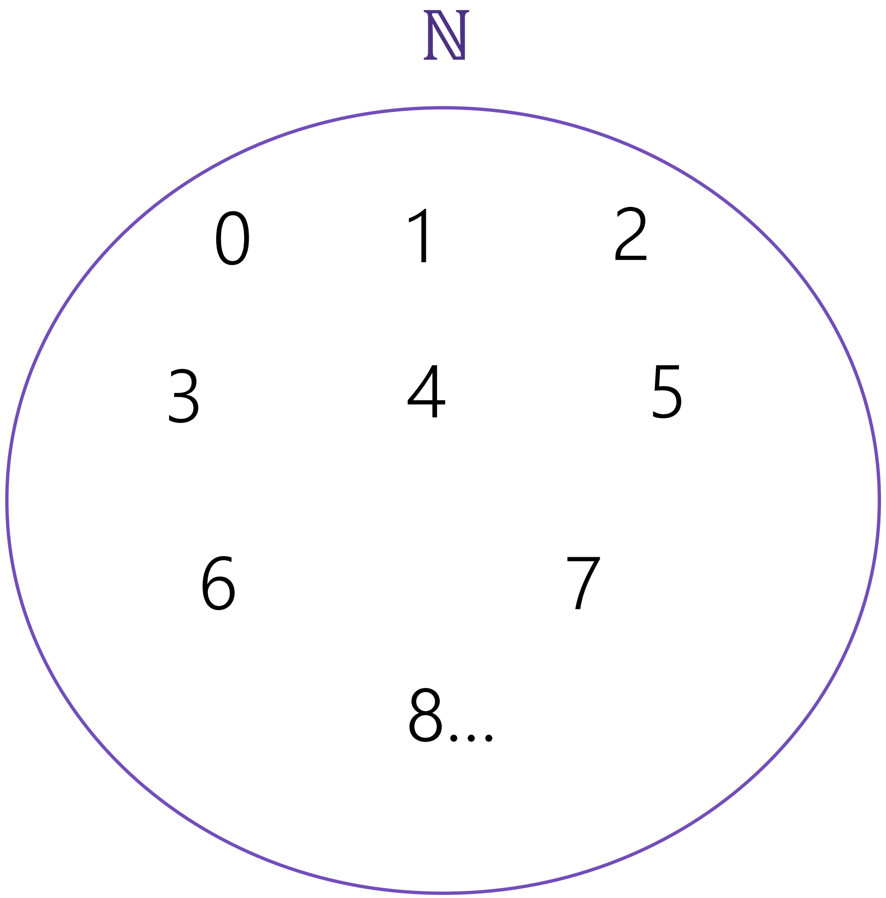
Basis: $6 \in S, 15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$.

We proved:

Base Case: P(6) and P(15)

IH → IS: If P(x) and P(y), then P(x+y)

Weak Induction is a special case of Structural



Basis: $\mathbf{0} \in \mathbb{N}$
Recursive: if $k \in \mathbb{N}$ then $k+1 \in \mathbb{N}$.

We proved:

Base Case: P(0)

IH → IS: If P(k), then P(k+1)

Strings

Σ will be an **alphabet** the set of all the letters you can use in strings.

E.g. $\Sigma = \{0,1\}$

E.g. $\Sigma = \{a, b, c, \dots, z, \_\}$

Σ^* is the set of all **strings** you can build off of the letters.

E.g. if $\Sigma = \{0,1\}$, then $01001 \in \Sigma^*$

E.g. if $\Sigma = \{a, b, c, \dots, z, \_\}$, then $i\_love\_recursive\_sets \in \Sigma^*$

ε is the **empty string**

Analogous to “” in Java

Strings

The set of all strings Σ^* can be defined as:

Basis Step: $\varepsilon \in \Sigma^*$.

Recursive Step: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

wa means the string of w with the character a appended

Functions on Strings

Length:

$\mathrm{len}(\varepsilon) = 0$

$\mathrm{len}(wa) = \mathrm{len}(w) + 1$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$;
$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Number of $\boldsymbol{c}$'s in a string

$\#_c(\varepsilon) = 0$
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$;
$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma \setminus \{c\}$.

Claim: len(x·y)=len(x) + len(y) for all $x, y \in \Sigma^*$.

Let $P(y)$ be "$\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$." Prove $P(y)$ for all $y \in \Sigma^*$.

Notice the strangeness of this $P()$. There is a "for all x" inside the definition of $P(y)$.

That means we'll have to introduce an arbitrary x as part of the base case and the inductive step!

Basis: $\varepsilon \in \Sigma^*$.
Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

Claim: len(x·y)=len(x) + len(y) for all $x, y \in \Sigma^*$.

1. Introduction:

2. Base Case:

3. Inductive Hypothesis:

4. Inductive Step:

Basis: $\varepsilon \in \Sigma^*$.
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1. Introduction: Let $P(y)$ be "$\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$." We prove $P(y)$ for all $y \in \Sigma^*$ by structural induction.

2. Base Case:

3. Inductive Hypothesis:

4. Inductive Step:

5. Therefore $P(y)$ holds for all $y \in \Sigma^*$by the principle of induction.

Basis: $\varepsilon \in \Sigma^*$.
Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

Claim: len(x·y)=len(x) + len(y) for all $x, y \in \Sigma^*$.

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2. Base Case: Let x be an arbitrary string. Then $\text{len}(x \cdot \epsilon) = \text{len}(x) = \text{len}(x) + 0 = \text{len}(x) + \text{len}(\varepsilon)$

3. Inductive Hypothesis:

4. Inductive Step:

5. Therefore $P(y)$ holds for all $\text{y} \in \Sigma^*$by the principle of induction.

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3. Inductive Hypothesis: Suppose $P(w)$ for an arbitrary $w \in \Sigma^*$.

4. Inductive Step: Let $x \in \Sigma^*$be an arbitrary string, and $a \in \Sigma$ be arbitrary.

$\text{len}(xwa) = \text{len}(xw) + 1$ (by definition of len)

$= \text{len}(x) + \text{len}(w) + 1$ (by IH)

$= \text{len}(x) + \text{len}(wa)$ (by definition of len)

Therefore, $\text{len}(xwa) = \text{len}(x) + \text{len}(wa)$.

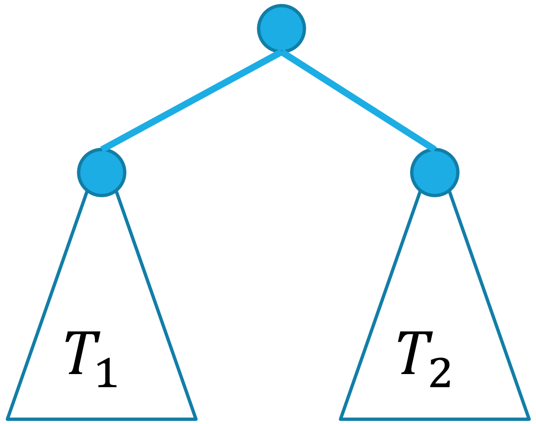
5. Therefore $P(y)$ holds for all y $\in \Sigma^*$by the principle of induction.

More Structural Sets

Binary Trees are another common source of structural induction.

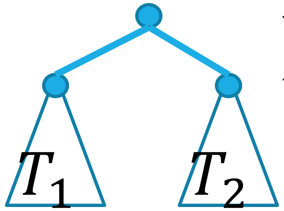
Basis Step: A single node is a rooted binary tree. 

Recursive Step: If T_1 and T_2 are rooted binary trees with roots r_1 and r_2, then a tree rooted at a new node, with children r_1, r_2 is a binary tree.



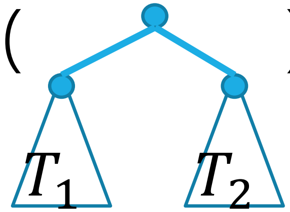
Functions on Binary Trees

size()=1

size(

) = size(T_1) + size(T_2) + 1

height() = 0

height(

) = 1+max(height(T_1),height(T_2))

Structural Induction on Binary Trees

For every rooted binary tree T, size(T) $\leq 2^{height(T)+1} - 1$

We'll show this next time.