Even More Induction

CSE 311 Winter 2023
Lecture 16

Let’s Try Another! Stamp Collecting

I have 4 cent stamps and 5 cent stamps (as many as I want of each). Prove that I can make exactly n cents worth of stamps for all $n \geq 12$.

Try for a few values.

Then think…how would the inductive step go?

Stamp Collection, Done Wrong

Define $P(n)$ I can make n cents of stamps with just 4 and 5 cent stamps.

We prove $P(n)$ is true for all $n \geq 12$ by induction on n.

Base Case:

12 cents can be made with three 4 cent stamps.
Inductive Hypothesis Suppose $P(k)$, $k \geq 12$.

Inductive Step:

We want to make $k+1$ cents of stamps. By IH we can make k cents exactly with stamps. Replace one of the 4 cent stamps with a 5 cent stamp.

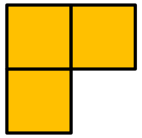
$P(n)$ holds for all n by the principle of induction.

Stamp Collection, Done Wrong

What if the starting point doesn't have any 4 cent stamps?

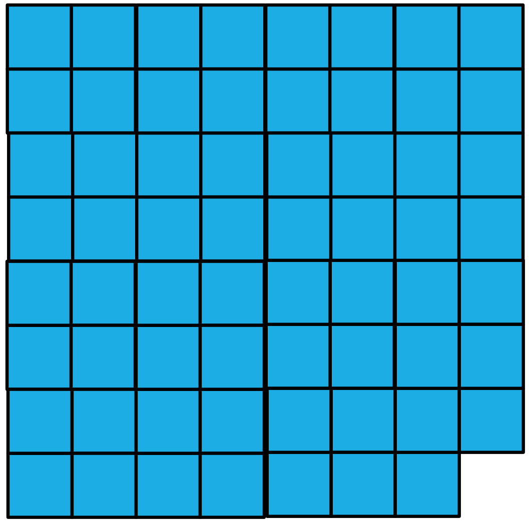
Like, say, 15 cents = 5+5+5.

Gridding



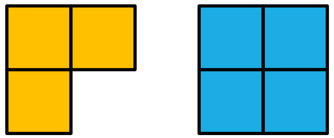
I've got a bunch of these 3 piece tiles.

I want to fill a 2^nx2^n grid ($n \geq 1$) with the pieces, except for a 1x1 spot in a corner.



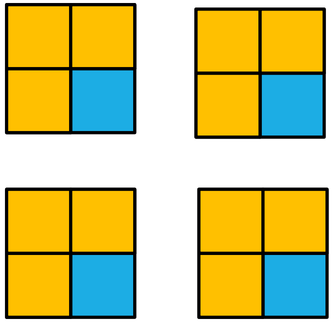
Gridding: Not a formal proof, just a sketch

Base Case: $n = 1$



Inductive hypothesis: Suppose you can tile a 2^kx2^k grid, except for a corner.

Inductive step: 2^{k+1}x2^{k+1}, divide into quarters. By IH can tile…



Recursively Defined Functions

Just like induction works will with recursive code, it also works well for recursively-defined functions.

Define the Fibonacci numbers as follows:

$f(0) = 1$

$f(1) = 1$

$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

*This is a somewhat unusual definition, $f(0) = 0, f(1) = 1$ is more common.

Fibonacci Inequality

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

$$f(0) = 1; \qquad f(1) = 1$$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Fibonacci Inequality

$f(0) = 1; \qquad f(1) = 1$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Show that $f(n) \leq 2^n$ for all $n \geq 0$ by induction.

Define $P(n)$ to be "$f(n) \leq 2^n$" We show $P(n)$ is true for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0)$: $f(0) = 1 \leq 1 = 2^0$.

$(n = 1)$: $f(1) = 1 \leq 2 = 2^1$.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 1$.

Inductive step:

Target: $P(k+1)$. i.e. $f(k+1) \leq 2^{k+1}$

Fibonacci Inequality

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Base Cases: $(n = 0)$: $f(0) = 1 \leq 1 = 2^0$.

$(n = 1)$: $f(1) = 1 \leq 2 = 2^1$.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 1$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have $f(k+1) \leq 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$.

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

[Define $P(n)$]

Base Case

Inductive Hypothesis

Inductive Step

[conclusion]

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$3|(2^{2n}-1)$." We show $P(n)$ holds for all $n \in \mathbb{N}$.

Base Case ($n = 0$) note that $2^{2n} - 1 = 2^0 - 1 = 0$. Since $3 \cdot 0 = 0$, and 0 is an integer, $3|(2^{2\cdot 0}-1)$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step:

Target: $P(k+1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

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Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$

Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$

FORCE the expression in your IH to appear

Target: $P(k+1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

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Inductive Step: By inductive hypothesis, $3|(2^{2k}-1)$. i.e. there is an integer j such that $3j = 2^{2k} - 1$.

$2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4\left(2^{2k} - 1\right) + 4 - 1$

By IH, we can replace $2^{2k} - 1$ with $3j$ for an integer j

$2^{2(k+1)} - 1 = 4(3j) + 4 - 1 = 3(4j) + 3 = 3(4j + 1)$

Since $4j + 1$ is an integer, we meet the definition of divides and we have:

Target: $P(k + 1)$, i.e. $3|(2^{2(k+1)}-1)$

Therefore, we have $P(n)$ for all $n \in \mathbb{N}$ by the principle of induction.

Claim: $3|(2^{2n}-1)$ for all $n \in \mathbb{N}$.

That inductive step might still seem like magic.

It sometimes helps to run through examples, and look for patterns:

$2^{2\cdot 0} - 1 = 0 = 3 \cdot 0$

$2^{2\cdot 1} - 1 = 3 = 3 \cdot 1$

$2^{2\cdot 2} - 1 = 15 = 3 \cdot 5$

$2^{2\cdot 3} - 1 = 63 = 3 \cdot 21$

$2^{2\cdot 4} - 1 = 255 = 3 \cdot 85$

$2^{2\cdot 5} - 1 = 1023 = 3 \cdot 341$

The divisor goes from k to $4k + 1$

$0 \to 4 \cdot 0 + 1 = 1$

$1 \to 4 \cdot 1 + 1 = 5$

$5 \to 4 \cdot 5 + 1 = 21$

…

That might give us a hint that $4k + 1$ will be in the algebra somewhere, and give us another intermediate target.

Induction: Hats!

You have n people in a line ($n \geq 2$). Each of them wears either a purple hat or a gold hat. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that for every arrangement of the line satisfying the rule above, there is a person with a purple hat next to someone with a gold hat.

Yes this is kinda obvious. I promise this is good induction practice.

Yes you could argue this by contradiction. I promise this is good induction practice.

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have $P(n)$ for all $n \geq 2$

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We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

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Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2:. There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length k, has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have $P(k + 1)$.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Fibonacci Inequality Two

$$f(0) = 1; \qquad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

[Define $P(n)$]

Base Cases:

Inductive Hypothesis:

Inductive step:

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

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Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

Target: $f(k+1) \geq 2^{(k+1)/2}$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

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$f(k+1) \geq 2^{k/2} + 2^{(k-1)/2}$

$\geq 2^{(k+1)/2}$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

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Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

$$\begin{aligned} f(k+1) &\geq 2^{k/2} + 2^{(k-1)/2} \\ &= 2^{(k-1)/2}\left(\sqrt{2} + 1\right) \\ &\geq 2^{(k-1)/2} \cdot 2 \\ &\geq 2^{(k+1)/2} \end{aligned}$$

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

More Practice

Even More Induction Practice

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Claim: $h(n) \geq g(n)$ for all integers $n \geq 1$

Even More Induction Practice

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case

Inductive Hypothesis:

Inductive Step:

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

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We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$g(k+1) = (k+1) \cdot g(k)$

$= (k+1)^{k+1}$.

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

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Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$
\begin{aligned}
g(k+1) &= (k+1) \cdot g(k) \\
&\leq (k+1) \cdot h(k) \text{ by IH.} \\
\\
\\
&= (k+1)^{k+1}.
\end{aligned}
$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

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Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

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Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$
\begin{aligned}
g(k+1) &= (k+1) \cdot g(k) \\
&\leq (k+1) \cdot h(k) && \text{by IH.} \\
&\leq (k+1) \cdot k^k && \text{by definition of } h(k) \\
&\leq (k+1) \cdot (k+1)^k \\
&= (k+1)^{k+1}.
\end{aligned}
$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice

Define $P(n)$ to be "$\mathrm{h}(n) \geq g(n)$ for all integers $n \geq 1$

We show $P(n)$ for all $n \geq 1$ by induction on n.

Base Case ($n = 1$): $h(n) = 1^1 = 1 \geq 1 = 1 \cdot 1 = 1 \cdot g(0) = g(1)$.

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 1$.

Inductive Step:

$$\begin{aligned} g(k+1) &= (k+1) \cdot g(k) \\ &\leq (k+1) \cdot h(k) && \text{by IH.} \\ &\leq (k+1) \cdot k^k && \text{by definition of } h(k) \\ &\leq (k+1) \cdot (k+1)^k \\ &= (k+1)^{k+1}. \end{aligned}$$

Thus $P(k+1)$ holds.

Therefore, we have $P(n)$ for all $n \geq 1$ by induction on n.

Let $g(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot g(n-1) & \text{otherwise} \end{cases}$

Let $h(n) = n^n$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$):

Inductive Hypothesis:

Inductive Step:

[Conclusion]

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

Target: $\sum_{i=0}^{k+1} 2 + 3i = \frac{([\mathrm{k}+1]+1)(3[\mathrm{k}+1]+4)}{2}$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2+3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2+3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$\sum_{i=0}^{k+1} 2+3i = \left(\sum_{i=0}^{k} 2+3i\right) + \big(2+3(k+1)\big)$. By IH, we have:

$\sum_{i=0}^{k+1} 2+3i = \frac{(\mathrm{k}+1)(3\mathrm{k}+4)}{2} + 2 + 3\mathrm{k} + 3 = ????$

$$= \frac{([\mathrm{k}+1]+1)(3[\mathrm{k}+1]+4)}{2}$$

Even More Induction Practice: Sums

Let $P(n)$ be $\sum_{i=0}^{n} 2 + 3i = \frac{(n+1)(3n+4)}{2}$

Show $P(n)$ for all $n \in \mathbb{N}$ by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{0} 2 + 3i = 2 = \frac{4}{2} = \frac{(0+1)(3\cdot 0+4)}{2}$

Inductive Hypothesis: Suppose $P(k)$ is true for an arbitrary $k \geq 0$.

Inductive Step:

$\sum_{i=0}^{k+1} 2 + 3i = (\sum_{i=0}^{k} 2 + 3i) + \left(2 + 3(k+1)\right)$. By IH, we have:

$\sum_{i=0}^{k+1} 2 + 3i = \frac{(\mathrm{k}+1)(3\mathrm{k}+4)}{2} + 2 + 3\mathrm{k} + 3 = \frac{3\mathrm{k}^2+7\mathrm{k}+4}{2} + \frac{6\mathrm{k}+10}{2} = \frac{3\mathrm{k}^2+13\mathrm{k}+14}{2} = \frac{(3\mathrm{k}+7)(\mathrm{k}+2)}{2} = \frac{([\mathrm{k}+1]+1)(3[\mathrm{k}+1]+4)}{2}$

Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.