Wrap-up Number Theory

CSE 311 Winter 2023
Lecture 15

Announcements

If you turn in HW5 part 2 today, we're hoping to get you feedback on HW5 part 2 by Sunday morning.

If you turn in HW5 part 2 using late days, we're hoping to get you feedback by sometime Sunday afternoon.

HW5 part 2 solutions will be posted on Ed Saturday morning.

There's an assignment "NOT the real midterm" on gradescope so you can see what the midterm will look like logistically.

Announcements

We designed the midterm to take 30 minutes.

But you have 2 hours (of your choice) **from when you open it**.

Open notes, open internet; but no discussion with other students.

We'll release it at 4:30 PM on Friday, due at 11:59 PM on Sunday.

Starting 4:30 PM Friday, we'll only answer private questions on Ed. And only "clarification" questions.

HW6 will come out early next week, due on the 18^{th}.

Announcements

What’s up with this “it’s supposed to be 30 minutes, but you have 2 hours thing?”

You’ve never done time-constrained proof writing before. Use this as a practice run (could you have gotten close to 30 minutes? If not, do you have to change something for the final, if it’s in-person?)

When we gave take-home exams in prior quarters, they often dragged out for days for students, with the last hours being (not particularly useful/educational) polishing of solutions.

We know you’re time-constrained, you should still polish your answers, but we understand you’re more limited than on homework.

Announcements

HW4 was harder than the last few. That's normal.

It's easy to get frustrated at this point of the class, you just got back a hard homework, you've started on one of our toughest concepts (induction), and we're about to have a midterm.

Don't check-out! Looking at grades/feedback is never fun, but it's a really key way to learn.

Plan For Today

Nothing “new” today, except the gcd,lcm notation, will show up on the midterm or homework.

Our goals are:

1. See that number theory results can make code faster in unexpected ways.

2. See a bit of code analysis (a preview of 332).

3. Hopefully say “oh neat, I understand a *little bit* about how secure online communication works”

You should not expect to fully understand *anything* from today.

We’re going to skip a bunch of slides today. If you’re interested read them for fun. If not, then skip them!

GCD and LCM

Greatest Common Divisor

The Greatest Common Divisor of a and $\boldsymbol{b}$ (gcd(a,b)) is the largest integer $\boldsymbol{c}$ such that $\boldsymbol{c|a}$ and $\boldsymbol{c|b}$

Least Common Multiple

The Least Common Multiple of a and $\boldsymbol{b}$ (lcm(a,b)) is the smallest positive integer $\boldsymbol{c}$ such that $\boldsymbol{a|c}$ and $\boldsymbol{b|c}$.

Try a few values…

gcd(100,125)

gcd(17,49)

gcd(17,34)

gcd(13,0)

lcm(7,11)

lcm(6,10)

How do you calculate a gcd?

You could:

Find the prime factorization of each

Take all the common ones. E.g.

gcd(24,20)=gcd($2^3 \cdot 3, 2^2 \cdot 5) = 2$^{$\min(2,3)$} $= 2$^2 $= 4$.

(lcm has a similar algorithm – take the maximum number of copies of everything)

But that's….really expensive. Mystery finds gcd.

```
public int Mystery(int m, int n){
     if(m<n){
          int temp = m;
          m=n;
          n=temp;
     }
     while(n != 0) {
          int rem = m % n;
          m=n;
          n=temp;
     }
     return m;
}
```

GCD fact

If a and b are positive integers, then gcd(a,b) = gcd(b, a % b)

Why is this true? The proof isn't easy, it's at the end of this deck.

Why should you care?

So…what's it good for?

Remember everything we're learning contributes to us eventually understanding RSA.
This is a key step in generating keys.

Suppose I want to solve $7x \equiv 1 (mod\ n)$

Just multiply both sides by $\frac{1}{7}$…

Oh wait. We want a number to multiply by 7 to get 1.

What number can we pick?

The next two slides are going to get more abstract…we're listing out the facts we need to solve that equation.

Bézout's Theorem

Bézout's Theorem

If a and b are positive integers, then there exist integers s and t such that
gcd(a,b)$= sa + tb$

We're not going to prove this theorem…

But it turns out `Mystery` can be extended to find them.

So…what's it good for?

Suppose I want to solve $7x \equiv 1(mod\ n)$

Just multiply both sides by $\frac{1}{7}$…

Oh wait. We want a number to multiply by $\mathbf{7}$ to get $\mathbf{1}$.

If the gcd(7,n) = 1

Then $s \cdot 7 + tn = 1$, so $7s - 1 = -tn$ i.e. $n|(7s - 1)$ so $7s \equiv 1(mod\ n)$.

So the s from Bézout's Theorem is what we should multiply by!

Ok…how am I supposed to find s, t?

It turns out that while you're calculating the gcd (using the Mystery algorithm), you can keep some extra information recorded, and end up with the s, t

This is called the "extended Euclidian algorithm"

Examples in these slides.

Proving the key fact about gcds

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that y is a common divisor of a and b.

By definition of gcd, $y|b$ and $y|(a\%b)$. So it is enough to show that $y|a$.

Applying the definition of divides we get $b = yk$ for an integer k, and $(a\%b) = yj$ for an integer j.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q.

Plugging in both of our other equations:

$a = qyk + yj = y(qk + j)$. Since q, k, and j are integers, $y|a$. Thus y is a common divisor of a, b and thus $y \leq x$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

By definition of gcd, $\mathrm{x}|b$ and $x|a$. So it is enough to show that $\mathrm{x}|(a\%b)$.

Applying the definition of divides we get $b = xk'$ for an integer k', and $\mathrm{a} = xj'$ for an integer j'.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q

Plugging in both of our other equations:

$xj' = qxk' + a\%b$. Solving for $a\%b$, we have $a\%b = xj' - qxk' = x(j' - qk')$. So $x|(a\%b)$. Thus x is a common divisor of $b, a\%b$ and thus $x \leq y$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

We have shown $x \leq y$ and $y \leq x$.

Thus $x = y$, and $\gcd(a, b) = \gcd(b, a\%b)$.

Euclidian Algorithm

Euclid’s Algorithm

```
while(n != 0) {
        int rem = m % n;
        m=n;
        n=rem;
    }
```

gcd(660,126)

Euclid's Algorithm

```
while(n != 0) {
        int rem = m % n;
        m=n;
        n=rem;
    }
```

$$
\begin{aligned}
\gcd(660,126) &= \gcd(126, 660 \bmod 126) &&= \gcd(126, 30)\\
&= \gcd(30, 126 \bmod 30) &&= \gcd(30, 6)\\
&= \gcd(6, 30 \bmod 6) &&= \gcd(6, 0)\\
&= 6
\end{aligned}
$$

Tableau form

$$
\begin{aligned}
\boxed{660} &= 5 \cdot \boxed{126} + 30\\
126 &= 4 \cdot 30 + \textcircled{6}\\
30 &= 5 \cdot 6 + 0
\end{aligned}
$$

Starting Numbers

Final answer

Bézout's Theorem

Bézout's Theorem

If a and b are positive integers, then there exist integers s and t such that
gcd(a,b)$= sa + tb$

We're not going to prove this theorem…

But we'll show you how to find s,t for any positive integers a, b.

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

gcd(35,27)

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

gcd(35,27) = gcd(27, 35%27) = gcd(27,8)
= gcd(8, 27%8) = gcd(8, 3)
= gcd(3, 8%3) = gcd(3, 2)
= gcd(2, 3%2) = gcd(2,1)
= gcd(1, 2%1) = gcd(1,0)

$$35 = 1 \cdot 27 + 8$$
$$27 = 3 \cdot 8 + 3$$
$$8 = 2 \cdot 3 + 2$$
$$3 = 1 \cdot 2 + 1$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$
\begin{aligned}
35 &= 1 \cdot 27 + 8 \\
27 &= 3 \cdot 8 + 3 \\
8 &= 2 \cdot 3 + 2 \\
3 &= 1 \cdot 2 + 1
\end{aligned}
$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$
\begin{aligned}
35 &= 1 \cdot 27 + 8 \\
27 &= 3 \cdot 8 + 3 \\
8 &= 2 \cdot 3 + 2 \\
3 &= 1 \cdot 2 + 1
\end{aligned}
$$

$$
\begin{aligned}
8 &= 35 - 1 \cdot 27 \\
3 &= 27 - 3 \cdot 8 \\
2 &= 8 - 2 \cdot 3 \\
1 &= 3 - 1 \cdot 2
\end{aligned}
$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$
\begin{aligned}
8 &= 35 - 1 \cdot 27 \\
3 &= 27 - 3 \cdot 8 \\
2 &= 8 - 2 \cdot 3 \\
1 &= 3 - 1 \cdot 2
\end{aligned}
$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$
\begin{aligned}
8 &= 35 - 1 \cdot 27 \\
3 &= 27 - 3 \cdot 8 \\
2 &= 8 - 2 \cdot 3 \\
1 &= 3 - 1 \cdot 2
\end{aligned}
$$

$$
\begin{aligned}
1 &= 3 - 1 \cdot 2 \\
&= 3 - 1 \cdot (8 - 2 \cdot 3) \\
&= -1 \cdot 8 + 2 \cdot 3
\end{aligned}
$$

Extended Euclidian Algorithm

Step 1 compute gcd(a,b); keep tableau information.

Step 2 solve all equations for the remainder.

Step 3 substitute backward

$$8 = 35 - 1 \cdot 27$$
$$3 = 27 - 3 \cdot 8$$
$$2 = 8 - 2 \cdot 3$$
$$1 = 3 - 1 \cdot 2$$

gcd(27,35) = $13 \cdot 27 + (-10) \cdot 35$

$$\begin{aligned} 1 &= 3 - 1 \cdot 2 \\ &= 3 - 1 \cdot (8 - 2 \cdot 3) \\ &= -1 \cdot 8 + 3 \cdot 3 \\ &= -1 \cdot 8 + 3(27 - 3 \cdot 8) \\ &= 3 \cdot 27 - 10 \cdot 8 \\ &= 3 \cdot 27 - 10(35 - 1 \cdot 27) \\ &= 13 \cdot 27 - 10 \cdot 35 \end{aligned}$$

When substituting back, you keep the larger of m, n and the number you just substituted. Don't simplify further! (or you lose the form you need)

Try it

Solve the equation $7y \equiv 3(mod\ 26)$

What do we need to find?

The multiplicative inverse of $7(\mathbf{mod}\ 26)$

Finding the inverse…

gcd(26,7) = gcd(7, 26%7) = gcd(7,5)

= gcd(5, 7%5) = gcd(5,2)

= gcd(2, 5%2) = gcd(2, 1)

= gcd(1, 2%1) = gcd(1,0)= 1.

$$1 = 5 - 2 \cdot 2$$
$$= 5 - 2(7 - 5 \cdot 1)$$
$$= 3 \cdot 5 - 2 \cdot 7$$
$$= 3 \cdot (26 - 3 \cdot 7) - 2 \cdot 7$$
$$3 \cdot 26 - 11 \cdot 7$$

-11 is a multiplicative inverse.
We'll write it as 15, since we're working mod 26.

$$26 = 3 \cdot 7 + 5\ ;\ 5 = 26 - 3 \cdot 7$$

$$7 = 5 \cdot 1 + 2\ ;\ 2 = 7 - 5 \cdot 1$$

$$5 = 2 \cdot 2 + 1\ ;\ 1 = 5 - 2 \cdot 2$$

Try it

Solve the equation $7y \equiv 3(mod\ 26)$

What do we need to find?
The multiplicative inverse of $7\ (mod\ 26)$.

$15 \cdot 7 \cdot y \equiv 15 \cdot 3(mod\ 26)$

$y \equiv 45(mod\ 26)$

Or $y \equiv 19(mod\ 26)$

So $26|19 - y$, i.e. $26k = 19 - y$ (for $k \in \mathbb{Z}$) i.e. $y = 19 - 26 \cdot k$ for any $k \in \mathbb{Z}$

So $\{\dots, -7, 19, 45, \dots 19 + 26k, \dots\}$

Key Steps in RSA

Given two numbers, we can find their gcd quickly.

If we have an equation

$ax \equiv b \pmod n$

And $\gcd(a, n) = 1$ then we can quickly find a number to multiply the equation by to solve for x.

RSA Encryption

Framing Device

We're going to give you enough background to (mostly) understand the RSA encryption system.

Key generation [edit]

The keys for the RSA algorithm are generated in the following way:

1. Choose two distinct prime numbers p and q.
   - For security purposes, the integers p and q should be chosen at random and should be similar in magnitude but differ in length by a few digits to make factoring harder.[2] Prime integers can be efficiently found using a primality test.
   - p and q are kept secret.
2. Compute $n = pq$.
   - n is used as the modulus for both the public and private keys. Its length, usually expressed in bits, is the key length.
   - n is released as part of the public key.
3. Compute $\lambda(n)$, where λ is Carmichael's totient function. Since $n = pq$, $\lambda(n) = \mathrm{lcm}(\lambda(p), \lambda(q))$, and since p and q are prime, $\lambda(p) = \varphi(p) = p - 1$, and likewise $\lambda(q) = q - 1$. Hence $\lambda(n) = \mathrm{lcm}(p - 1, q - 1)$.
   - $\lambda(n)$ is kept secret.
   - The lcm may be calculated through the Euclidean algorithm, since $\mathrm{lcm}(a, b) = |ab|/\gcd(a, b)$.
4. Choose an integer e such that $1 < e < \lambda(n)$ and $\gcd(e, \lambda(n)) = 1$; that is, e and $\lambda(n)$ are coprime.
   - e having a short bit-length and small Hamming weight results in more efficient encryption – the most commonly chosen value for e is $2^{16} + 1 = 65\,537$. The smallest (and fastest) possible value for e is 3, but such a small value for e has been shown to be less secure in some settings.[15]
   - e is released as part of the public key.
5. Determine d as $d \equiv e^{-1} \pmod{\lambda(n)}$; that is, d is the modular multiplicative inverse of e modulo $\lambda(n)$.
   - This means: solve for d the equation $d \cdot e \equiv 1 \pmod{\lambda(n)}$; d can be computed efficiently by using the extended Euclidean algorithm, since, thanks to e and $\lambda(n)$ being coprime, said equation is a form of Bézout's identity, where d is one of the coefficients.
   - d is kept secret as the *private key exponent*.

The *public key* consists of the modulus n and the public (or encryption) exponent e. The *private key* consists of the private (or decryption) exponent d, which must be kept secret. p, q, and $\lambda(n)$ must also be kept secret because they can be used to calculate d. In fact, they can all be discarded after d has been computed.[16]

Framing Device

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Key generation [edit]

Prime Numbers

The keys for the RSA algorithm are genera

1. Choose two distinct prime numbers *p* and *q*.
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   - *p* and *q* are kept secret.

Modular Arithmetic

2. Compute $n = pq$.
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3. Compute $\lambda(n)$, where λ is Carmichael's totient function. Since $n = pq$, $\lambda(n) = \mathrm{lcm}(\lambda(p), \lambda(q))$, and since *p* and *q* are prime, $\lambda(p) = \varphi(p) = p - 1$, and likewise $\lambda(q) = q - 1$. Hence $\lambda(n) = \mathrm{lcm}(p - 1, q - 1)$.
   - $\lambda(n)$ is kept secret.
   - The lcm may be calculated through the Euclidean algorithm, since lcm(*a*, *b*)

Modular Multiplicative Inverse

4. Choose an integer *e* such that $1 < e < \lambda(n)$ and $\gcd(e, \lambda(n)) = 1$; that is, *e* and $\lambda(n)$ are cop
   - *e* having a short bit-length and small Hamming weight results in more efficient en … e most commonly chosen value for *e* is $2^{16} + 1 = 65\,537$. The smallest (and fastest) possible value for *e* is 3, but such a small value for *e* has been shown to be less secure in some settings.[15]
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Bezout’s Theorem

5. Determine *d* as $d \equiv e^{-1} \pmod{\lambda(n)}$; that is, *d* is the modular multiplicative inverse of *e* modulo $\lambda(n)$.
   - This means: solve for *d* the equation $d \cdot e \equiv 1 \pmod{\lambda(n)}$; *d* can be computed efficiently by using the extended Euclidean algorithm, since, thanks to *e* and $\lambda(n)$ being coprime, said equation is a form of Bézout's identity, where *d* is one of the coefficients.
   - *d* is kept secret as the *private key exponent*.

Extended Euclidian Algorithm

The *public key* consists of the modulus *n* and the public (or encryption) exponent *e*. The *private key* consists of the private (or decryp … also be kept secret because they can be used to calculate *d*. In fact, they can all be discarded after *d* has been computed.[16]

Framing Device

We're going to give you enough background to (mostly) understand the RSA encryption system.

Encryption [edit]

After Bob obtains Alice's public key, he can send a message M to Alice.

To do it, he first turns M (strictly speaking, the un-padded plaintext) into an integer m (strictly speaking, the padded plaintext), such that $0 \le m < n$ by using an agreed-upon reversible protocol known as a padding scheme. He then computes the ciphertext c, using Alice's public key e, corresponding to

$$c \equiv m^e \pmod{n}.$$

This can be done reasonably quickly, even for very large numbers, using modular exponentiation. Bob then transmits c to Alice. Note that at least nine values of *m* will yield a ciphertext c equal to *m*,[22] but this is very unlikely to occur in practice.

Decryption [edit]

Alice can recover m from c by using her private key exponent d by computing

$$c^d \equiv (m^e)^d \equiv m \pmod{n}.$$

Given m, she can recover the original message M by reversing the padding scheme.

Framing Device

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Modular Exponentiation

Decryption [edit]

Alice can recover m from c by using her private key exponent d by computing

$$c^d \equiv (m^e)^d \equiv m \pmod{n}.$$

Given m, she can recover the original message M by reversing the padding scheme.

An application of all of this modular arithmetic

Amazon chooses random 512-bit (or 1024-bit) prime numbers p, q and an exponent e (often about 60,000).

Amazon calculates $\mathrm{n} = pq$. They tell your computer (n, e) (not p, q)

You want to send Amazon your credit card number a.

You compute $C = a^e \% n$ and send Amazon C.

Amazon computes d, the multiplicative inverse of e $(mod\ [p-1][q-1])$

Amazon finds $C^d \% n$

Fact: $a = C^d \% n$ as long as $0 < a < n$ and $p \nmid a$ and $q \nmid a$

How big are those numbers?

1230186684530117755130494958384962720772853569595334792197322
4521517264005072636575187452021997864693899564749427740638459
2519255732630345373154826850791702612214291346167042921431160
2221240479274737794080665351419597459856902143413

=

3347807169895689878604416984821269081770479498371376856891243
138898288379387800228761471165253174308773781446799948 9

×

3674604366679959042824463379962795263227915816434308764267603
2283815739666511279233373417143396810270092798736308917

How do we accomplish those steps?

That fact? You can prove it in the extra credit problem on HW5. It's a nice combination of lots of things we've done with modular arithmetic.

Let's talk about finding $C = a^e \% n$.

e is a BIG number (about 2^{16} is a common choice)

```
int total = 1;
for(int i = 0; i < e; i++){
    total = (a * total) % n;
}
```

Fast Exponentiation Algorithm

Let's build a faster algorithm.

Fast exponentiation – simple case. What if e is exactly 2^{16}?

```
int total = 1;
for(int i = 0; i < e; i++){
     total = a * total % n;
}
Instead:
int total = a;
for(int i = 0; i < log(e); i++){
     total = total^2 % n;
}
```

Fast Exponentiation Algorithm

What if e isn't exactly a power of 2?

Step 1: Write e in binary.

Step 2: Find $a^c \% n$ for c every power of 2 up to e.

Step 3: calculate a^e by multiplying a^c for all c where binary expansion of e had a 1.

Fast Exponentiation Algorithm

Find $4^{11}\%10$

Step 1: Write e in binary.

Step 2: Find $a^c\%n$ for c every power of 2 up to e.

Step 3: calculate a^e by multiplying a^c for all c where binary expansion of e had a 1.

Start with largest power of 2 less than e (8). 8's place gets a 1. Subtract power

Go to next lower power of 2, if remainder of e is larger, place gets a 1, subtract power; else place gets a 0 (leave remainder alone).

11 = 1011_2

Fast Exponentiation Algorithm

Find $4^{11}\%10$

Step 1: Write e in binary.

Step 2: Find $\boldsymbol{a^c\%n}$ for $\boldsymbol{c}$ every power of 2 up to $\boldsymbol{e}$.

Step 3: calculate a^e by multiplying a^c for all c where binary expansion of e had a 1.

$4^1\%10 = 4$

$4^2\%10 = 6$

$4^4\%10 = 6^2\%10 = 6$

$4^8\%10 = 6^2\%10 = 6$

Fast Exponentiation Algorithm

Find $4^{11}\%10$

Step 1: Write e in binary.

Step 2: Find $a^c\%n$ for c every power of 2 up to e.

Step 3: calculate $\boldsymbol{a^e}$ by multiplying $\boldsymbol{a^c}$ for all $\boldsymbol{c}$ where binary expansion of $\boldsymbol{e}$ had a $\mathbf{1}$.

$4^1\%10 = 4$

$4^2\%10 = 6$

$4^4\%10 = 6^2\%10 = 6$

$4^8\%10 = 6^2\%10 = 6$

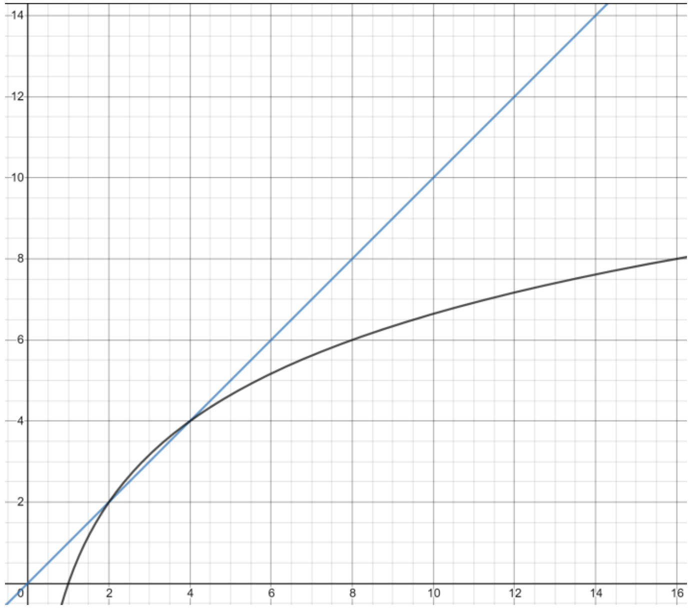
$$4^{11}\%10 = 4^{8+2+1}\%10 =$$
$$[(4^8\%10)\cdot\left(4^2\%10\right)\cdot(4\%10)]\%10 = (6\cdot 6\cdot 4)\%10$$
$$= (36\%10\cdot 4)\%10 = (6\cdot 4)\%10 = \ 24\%10 = 4.$$

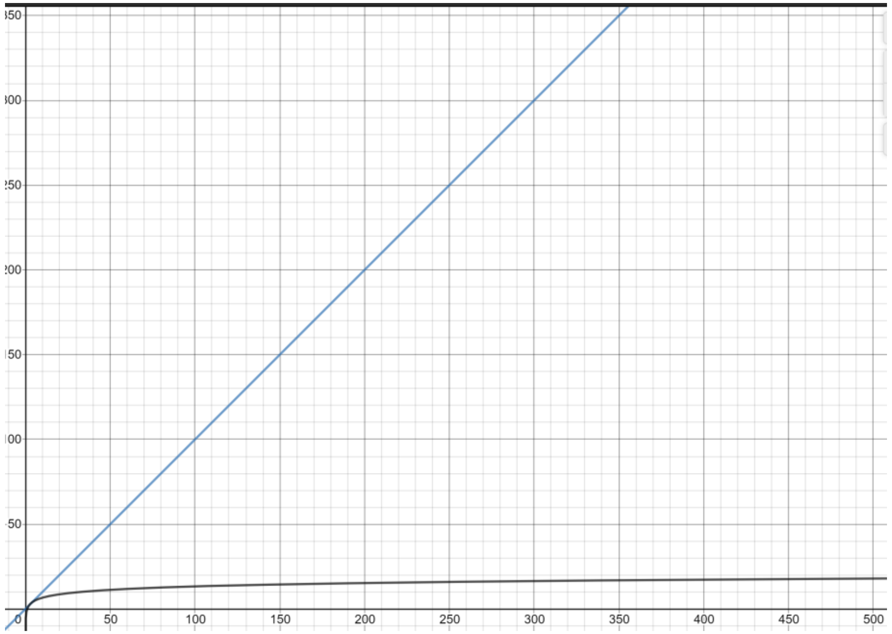
Fast Exponentiation Algorithm

Is it…actually fast?

The number of multiplications is between $\log_2 e$ and $2\log_2 e$.

That's A LOT smaller than e





One More Example for Reference

Find $3^{25}\%7$ using the fast exponentiation algorithm.

Find 25 in binary:

16 is the largest power of 2 smaller than 25. $(25 - 16) = 9$ remaining

8 is smaller than 9. $(9 - 8) = 1$ remaining.

4s place gets a 0.

2s place gets a $\mathbf{0}$

$1s$ place gets a 1

11001_2

One More Example for Reference

Find $3^{25}\%7$ using the fast exponentiation algorithm.

Find $3^{2^i}\%7$:

$3^1\%7 = 3$

$3^2\%7 = 9\%7 = 2$

$3^4\%7 = (3^2 \cdot 3^2)\%7 = (2 \cdot 2)\%7 = 4$

$3^8\%7 = \left(3^4 \cdot 3^4\right)\%7 = (4 \cdot 4)\%7 = 2$

$3^{16}\%7 = \left(3^8 \cdot 3^8\right)\%7 = (2 \cdot 2)\%7 = 4$

One More Example for Reference

Find $3^{25}\%7$ using the fast exponentiation algorithm.

$3^1\%7 = 3$

$3^2\%7 = 2$

$3^4\%7 = 4$

$3^8\%7 = 2$

$3^{16}\%7 = 4$

$$3^{25}\%7 = 3^{16+8+1}\%7$$
$$= [(3^{16}\%7) \cdot (3^8\%7) \cdot (3^1\%7)]\%7$$
$$= [4 \cdot 2 \cdot 3]\%7$$
$$= (1 \cdot 3)\%7 = 3$$

A Brief Concluding Remark

Why does RSA work? i.e. why is my credit card number “secret”?

Raising numbers to large exponents (in mod arithmetic) and finding multiplicative inverses in modular arithmetic are things computers can do quickly.

But factoring numbers (to find p, q to get d) or finding an “exponential inverse” (not the real term) directly are not things computers can do quickly. At least as far as we know.

An application of all of this modular arithmetic

Amazon chooses random 512-bit (or 1024-bit) prime numbers p, q and an exponent e (often about 60,000).

Amazon calculates $\mathrm{n} = pq$. They tell your computer (n, e) (not p, q)

You want to send Amazon your credit card number a.

You compute $C = a^e \% n$ and send Amazon C.

Amazon computes d, the multiplicative inverse of e $(mod\ [p-1][q-1])$

Amazon finds $C^d \% n$

Fact: $a = C^d \% n$ as long as $0 < a < n$ and $p \nmid a$ and $q \nmid a$

Extra Practice!

Warm up

Equivalence in modular arithmetic

Let $a \in \mathbb{Z}, b \in \mathbb{Z}, n \in \mathbb{Z}$ and $n > 0$.
We say $a \equiv b \ (mod\ n)$ if and only if $n|(b - a)$

Show that $a \equiv b \ (mod\ n)$ if and only if $b \equiv a(mod\ n)$

Show that $a\%n$=$(a - n)\%n$ Where $b\%c$ is the unique r such that $b = kc + r$ for some integer k.

The Division Theorem

For every $a \in \mathbb{Z}$, $\boldsymbol{d} \in \mathbb{Z}$ with $\boldsymbol{d} > \mathbf{0}$
There exist *unique* integers q, r with $0 \leq r < d$ Such that $a = dq + r$

Warm up

Show that $a \equiv b\ (mod\ n)$ if and only if $b \equiv a(mod\ n)$

$a \equiv b(mod\ n) \leftrightarrow n|(b-a) \leftrightarrow nk = b - a(\text{for } k \in \mathbb{Z}) \leftrightarrow$

$n(-k) = a - b(\text{for } -\text{k} \in \mathbb{Z}) \leftrightarrow n|(a-b) \leftrightarrow b \equiv a(mod\ n)$

Show that $a\%n$=$(a-n)\%n$ Where $b\%c$ is the unique r such that $b = kc + r$ for some integer k.

By definition of %, $a = qn + (a\%n)$ for some integer q. Subtracting n,

$a - n = (q-1)n + (a\%n)$. Observe that $q - 1$ is an integer, and that this is the form of the division theorem for $(a-n)\%n$. Since the division theorem guarantees a unique integer, $(a-n)\%n = (a\%n)$

% and Mod

Other resources use mod to mean an operation (takes in an integer, outputs an integer). We will not in this course. mod only describes $\equiv$. It's not "just on the right hand side"

Define $a\%b$ to be "the r you get from the division theorem"
i.e. the integer r such that $0 \leq r < d$ and $a = bq + r$ for some integer q.

This is the "mod function"

I claim $a\%n = b\%n$ if and only if $a \equiv b (mod\ n)$.

How do we show and if-and-only-if?

$a\%n = b\%n$ if and only if $a \equiv b(mod\ n)$

Backward direction:

Suppose $a \equiv b(mod\ n)$

$a\%n = (b - nk)\%n = b\%n$

$a\%n = b\%n$ if and only if $a \equiv b (mod\ n)$

Backward direction:

Suppose $a \equiv b(mod\ n)$

$n | b - a$ so $nk = b - a$ for some integer k. (by definitions of mod and divides).

So $a = b - nk$

Taking each side $\%n$ we get:

$a\%n = (b - nk)\%n = b\%n$

Where the last equality follows from k being an integer and doing k applications of the identity we proved in the warm-up.

$a\%n = b\%n$ if and only if $a \equiv b(mod\ n)$

Show the forward direction:

If $a\%n = b\%n$ then $a \equiv b(mod\ n)$.

This proof is a bit different than the other direction.

Remember to work from top and bottom!!

Equivalence in modular arithmetic

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Forward direction:

Suppose $a\%n = b\%n$.

By definition of %, $a = kn + (a\%n)$ and $b = jn + (b\%n)$ for integers k, j

Isolating $a\%n$ we have $a\%n = a - kn$. Since $a\%n = b\%n$, we can plug into the second equation to get: $b = jn + (a - kn)$

Rearranging, we have $b - a = (j - k)n$. Since k, j are integers we have $n|(b - a)$.

By definition of mod we have $a \equiv b(mod\ n)$.

More Number Theory Proofs

Caution

To fit proofs on these slides, I skipped some of the boilerplate steps (e.g. introducing variables as arbitrary, including a conclusion)

Don’t skip those on your homework/midterm, please ☺

Warm up

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Modular arithmetic so far

For all integers a, b, c, d, n where $n > 0$:

If $a \equiv b (mod\ n)$ then $a + c \equiv a + c (mod\ n)$.

If $a \equiv b (mod\ n)$ then $ac \equiv bc\ (mod\ n)$.

$a \equiv b (mod\ n)$ if and only if $b \equiv a\ (mod\ n)$.

$a\%n = (a - n)\%n$.

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By definition of mod we have $a \equiv b(mod\ n)$.

Why does the Euclidian Algorithm Work?

Correctness of an algorithm

The key to the Euclidian Algorithm being correct is that each time through the loop, you don't change the gcd of the variables `m,n`.

To prove the code correct, you really want an induction proof (it's good practice to think about it!). The inductive step relies on the fact we stated but didn't prove:

```
gcd(a,b) = gcd(b, a%b).
```

Let's prove it!

GCD fact

If a and b are positive integers, then gcd(a,b) = gcd(b, a % b)

How do you show two gcds are equal?

Call $a = \text{gcd}(w, x), b = \text{gcd}(y, z)$

If $b|w$ and $b|x$ then b is a common divisor of w, x so $b \leq a$

If $a|y$ and $a|z$ then a is a common divisor of y, z, so $a \leq b$

If $a \leq b$ and $b \leq a$ then $a = b$

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that y is a common divisor of a and b.

By definition of gcd, $y|b$ and $y|(a\%b)$. So it is enough to show that $y|a$.

Applying the definition of divides we get $b = yk$ for an integer k, and $(a\%b) = yj$ for an integer j.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q.

Plugging in both of our other equations:

$a = qyk + yj = y(qk + j)$. Since q, k, and j are integers, $y|a$. Thus y is a common divisor of a, b and thus $y \leq x$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \mathrm{gcd}(a, b)$ and $y = \mathrm{gcd}(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

By definition of gcd, $\mathrm{x}|b$ and $x|a$. So it is enough to show that $\mathrm{x}|(a\%b)$.

Applying the definition of divides we get $b = xk'$ for an integer k', and $\mathrm{a} = xj'$ for an integer j'.

By definition of mod, $a\%b$ is $a = qb + (a\%b)$ for an integer q

Plugging in both of our other equations:

$xj' = qxk' + a\%b$. Solving for $a\%b$, we have $a\%b = xj' - qxk' = x(j' - qk')$. So $x|(a\%b)$. Thus x is a common divisor of $b, a\%b$ and thus $x \leq y$.

gcd(a,b) = gcd(b, a % b)

Let $\mathrm{x} = \gcd(a, b)$ and $y = \gcd(b, a\%b)$.

We show that x is a common divisor of b and $\mathrm{a}\%b$.

We have shown $x \leq y$ and $y \leq x$.

Thus $x = y$, and $\gcd(a, b) = \gcd(b, a\%b)$.