Sets

+Number Theory

CSE 311 Winter 2023
Lecture 10

Doing a Proof

$\forall x \forall y([\texttt{rational}(x) \wedge \texttt{rational}(y)] \rightarrow \texttt{rational}(xy))$

"The product of two rational numbers is rational."

DON'T just jump right in!

Look at the statement, make sure you know:

1. What every word in the statement means.
2. What the statement as a whole means.
3. Where to start.
4. What your target is.

Let's do another!

"The product of two rational numbers is rational."

Let x, y be arbitrary rational numbers.

Therefore, xy is rational.

Since x and y were arbitrary, we can conclude the product of two rational numbers is rational.

Let's do another!

"The product of two rational numbers is rational."

Let x, y be arbitrary rational numbers.

By the definition of rational, $x = a/b$, $y = c/d$ for integers a, b, c, d where $b \neq 0$ and $d \neq 0$.

Multiplying, $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Since integers are closed under multiplication, ac and bd are integers.

Moreover, $bd \neq 0$ because neither b nor d is 0. Thus xy is rational.

Since x and y were arbitrary, we can conclude the product of two rational numbers is rational.

Now You Try

The sum of two even numbers is even.

1. Write the statement in predicate logic.
2. Write an English proof.
3. If you have lots of extra time, try writing the symbolic proof instead.

Now You Try

The sum of two even numbers is even.

Even

An integer x is even if (and only if) there exists an integer z, such that $x = 2z$.

Pollev.com/robbie

Help me adjust my explanation!

Make sure you know:

1. What every word in the statement means.
2. What the statement as a whole means.
3. Where to start.
4. What your target is.

1. Write the statement in predicate logic.
2. Write an English proof.
3. If you have lots of extra time, try writing the symbolic proof instead.

Here's What I got.

$\forall x \forall y([\text{Even}(x) \wedge \text{Even}(y)] \rightarrow \text{Even}(x+y))$

Let x, y be arbitrary integers, and suppose x and y are even.

By the definition of even, $x = 2a, y = 2b$ for some integers a and b.

Summing the equations, $x + y = 2a + 2b = 2(a + b)$.

Since a and b are integers, $a + b$ is an integer, so $x + y$ is even by the definition of even.

Since x, y were arbitrary, we can conclude the sum of two even integers is even.

Why English Proofs?

Those symbolic proofs seemed pretty nice. Computers understand them, and can check them.

So what's up with these English proofs?

They're far easier for **people** to understand.

But instead of a computer checking them, now a human is checking them.

Today

Is a laundry list of definitions – everything you ever wanted to know about sets and a pinch of number theory.

we'll get to do a proof, hopefully.

Sets

A set is an **unordered** group of **distinct** elements.

We'll always write a set as a list of its elements inside {curly, brackets}.

Variable names are capital letters, with lower-case letters for elements.

$|A| = 2$. "The size of A is 2." or "A has cardinality 2."

$A = \{\text{curly, brackets}\}$

$B = \{0,5,8,10\} = \{5,0,8,10\} = \{0,0,5,8,10\}$

$C = \{0,1,2,3,4, \dots\}$

Sets

Some more symbols:

$a \in A$ ("a is in A" or "a is an element of A") means a is one of the members of the set.

For $B = \{0,5,8,10\}$, $0 \in B$.

$A \subseteq B$ (A is a subset of B) means every element of A is also in B.

For $A = \{1,2\}$, $B = \{1,2,3\}$ $A \subseteq B$

Try it!

Let $A = \{1,2,3,4,5\}$

$B = \{1,2,5\}$

Is $A \subseteq A$?

Is $B \subseteq A$?

Is $A \subseteq B$?

Is $\{1\} \in A$?

Is $1 \in A$?

Try it!

Let $A = \{1,2,3,4,5\}$

$B = \{1,2,5\}$

Is $A \subseteq A$? Yes!

Is $B \subseteq A$? Yes

Is $A \subseteq B$? No

Is $\{1\} \in A$? No

Is $1 \in A$? Yes

Sets

Be careful about these two operations:

If $A = \{1,2,3,4,5\}$

$\{1\} \subseteq A$, but $\{1\} \notin A$

$\in$ asks: is this item in that box?

$\subseteq$ asks: is everything in this box also in that box?

Set Builder Notation

Sometimes we want to give a property and say "everything with that property is in the set (and nothing else is in the set)."

$A = \{x : \texttt{Even}(x)\}$

"The set of all x such that x is even."

In general $\{variable : Condition(variable)\}$

Sometimes the colon is replaced with |

Definitions

$A \subseteq B$ ("A is a subset of B") iff every element of A is also in B.

$$A \subseteq B \equiv \forall x (x \in A \rightarrow x \in B)$$

$A = B$ ("A equals B") iff A and B have identical elements.

$$A = B \equiv \forall x (x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$$

Proof Skeleton

How would we show $A \subseteq B$?

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

Let x be an arbitrary element of A

…

So x is also in B.

Since x was an arbitrary element of A, we have that $A \subseteq B$.

Proof Skeleton

That wasn't a "new" skeleton! It's exactly what we did when we wanted to prove $\forall x(P(x) \to Q(x))$!

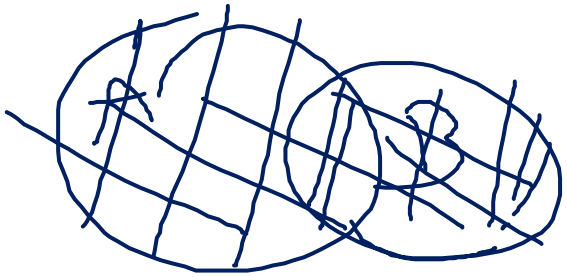
What about $A = B$?

$$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$$

Just do two subset proofs!

i.e. $\forall x(x \in A \to x \in B)$ and $\forall x(x \in B \to x \in A)$

What do we do with sets?



We combined propositions with ∨,∧, ¬.

We combine sets with ∩ [intersection],∪, [union] ¯[complement]

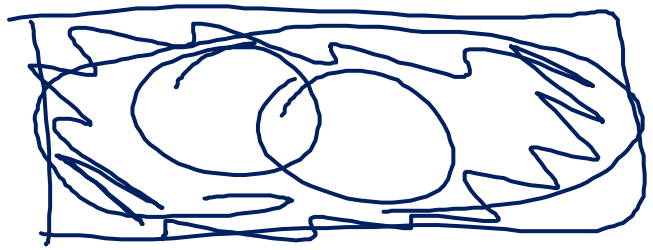
$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$\bar{A} = \{x : x \notin A\}$$

That's a lot of elements…if we take the complement, we'll have some "universe" $\mathcal{U}$, and $\bar{A} = \{x : x \in U \wedge x \notin A\}$
It's a lot like the domain of discourse.

A proof!



What's the analogue of DeMorgan's Laws…

$\bar{A} \cap \bar{B} = \overline{A \cup B}$

$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$

$\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$

A proof!

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$$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$$

$\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

…

That is, x is in the complement of $A \cup B$, as required.

Since x was arbitrary $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$

Let x be an arbitrary element of $\overline{A \cup B}$.

…

we get $x \in \bar{A} \cap \bar{B}$

Since x was arbitrary $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$

Since the subset relation holds in both directions, we have $\bar{A} \cap \bar{B} = \overline{A \cup B}$

A proof!

What's the analogue of DeMorgan's Laws...

$\bar{A} \cap \bar{B} = \overline{A \cup B}$

$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$

$\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$
Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.
By definition of $\cap$ $x \in \bar{A}$ and $x \in \bar{B}$. By definition of complement, $x \notin A \wedge x \notin B$.
Applying DeMorgan's Law, we get that it is not the case that $x \in A \vee x \in B$.
That is, x is in the complement of $A \cup B$, as required.
Since x was arbitrary $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$
Let x be an arbitrary element of $\overline{A \cup B}$.
By definition of complement, x is not an element of $A \cup B$. Applying the definition of union, we get, $\neg(x \in A \vee x \in B)$
Applying DeMorgan's Law, we get: $x \notin A \wedge x \notin B$
By definition of $\cap$ and complement, we get $x \in \bar{A} \cap \bar{B}$
Since x was arbitrary $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$

Since the subset relation holds in both directions, we have $\bar{A} \cap \bar{B} = \overline{A \cup B}$

Proof-writing advice

When you're writing a set equality proof, often the two directions are nearly identical, just reversed.

It's very tempting to use that $x \in A \leftrightarrow x \in B$ definition.

Be VERY VERY careful. It's easy to mess that up, at every step you need to be saying "if and only if."

Two claims, two proof techniques

Suppose I claim that for all sets A, B, C: $A \cap B \subseteq C$

That…doesn't look right.

How do you prove me wrong?

Two claims, two proof techniques

Suppose I claim that for all sets A, B, C: $A \cap B \subseteq C$

That…doesn't look right.

How do you prove me wrong?

Want to show: $\exists A, B, C: A \cap B \nsubseteq C$

Consider $A = \{1,2,3\}$, $B = \{1,2\}$, $C = \{2,3\}$, then $A \cap B = \{1,2\}$, which is not a subset of C.

Proof By [Counter]Example

To prove an existential statement (or disprove a universal statement), provide an example, and demonstrate that it is the needed example.

You don't have to explain where it came from! (In fact, you **shouldn't)**

Computer scientists and mathematicians like to keep an air of mystery around our proofs.

(or more charitably, we want to focus on just enough to believe the claim)

Skeleton of an Exists Proof

To show $\exists x(P(x))$

Consider x =[the value that will work]

[Show that x does cause $P(x)$ to be true.]

So [value] is the desired x.

You'll probably need some "scratch work" to determine what to set x to. That might not end up in the final proof!

Proof By Cases

Let $A = \{x : \texttt{Prime}(x)\}$, $B = \{x: \texttt{Odd}(x) \vee \texttt{PowerOfTwo}(x)\}$

Where $\texttt{PowerOfTwo}(x) \coloneqq \exists c(\texttt{Integer}(c) \wedge x = 2\texttt{\^}c)$

Prove $A \subseteq B$

We need two different arguments – one for 2 and one for all the other primes…

Proof By Cases

Let x be an arbitrary element of A.

We divide into two cases.

Case 1: x is even

If x is even and an element of A (i.e. both even and prime) it must be 2.
So it equals $2\text{^}c$ for $c = 1$, and thus is in B by definition of B.

Case 2: x is odd

Then $x \in B$ by satisfying the first requirement in the definition of B.

In either case, $x \in B$. Since an arbitrary element of A is also in B, we have $A \subseteq B$.

Proof By Cases

Make it clear how you decide which case your in.

It should be obvious your cases are “exhaustive”

Reach the same conclusion in each of the cases, and you can say you’ve got that conclusion no matter what (outside the cases).

Advanced version: sometimes you end up arguing a certain case “can’t happen”

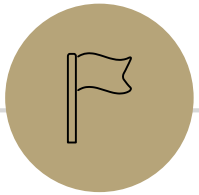
One More Set Operation

Given a set, let's talk about it's powerset.

$\mathcal{P}(A) = \{\mathrm{X}: \mathrm{X} \text{ is a subset of } A\}$

The powerset of A is the **set** of all subsets of A.

$\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$



Read on Your Own

Some old friends (and some new ones)

$\mathbb{N}$ is the set of **Natural Numbers; $\mathbb{N}$** = {0, 1, 2, …}
$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z}$ = {…, -2, -1, 0, 1, 2, …}
$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48
$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π,$\sqrt{2}$
[n] is the set **{1, 2, …, n}** when **n** is a positive integer
{} = ∅ is the **empty set**; the *only* set with no elements

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Our natural numbers start at 0.
Common in CS, other resources start at 1.

In LaTeX \mathbb{R}
In Office \doubleR

Use this symbol not {}.
In LaTex \varnothing In Office \emptyset.

More Connectors!

$A \setminus B$ "A minus B"

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$

$A \oplus B$ "XOR" (also called "symmetric difference")

$$A \oplus B = \{x : x \in A \oplus x \in B\}$$

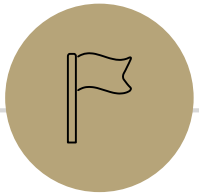
More Connectors!

$A \times B = \{(a, b) : a \in A \wedge b \in B\}$

Called "the Cartesian product" of A and B.

$\mathbb{R} \times \mathbb{R}$ is the "real plane" ordered pairs of real numbers.

$\{1,2\} \times \{1,2,3\} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$



Number Theory

Why Number Theory?

Applicable in Computer Science

"hash functions" (you'll see them in 332) commonly use modular arithmetic
Much of classical cryptography is based on prime numbers.

More importantly, a great playground for writing English proofs.

Framing Device

We're going to give you enough background to (mostly) understand the RSA encryption system.

Key generation [edit]

The keys for the RSA algorithm are generated in the following way:

1. Choose two distinct prime numbers p and q.
   - For security purposes, the integers p and q should be chosen at random and should be similar in magnitude but differ in length by a few digits to make factoring harder.[2] Prime integers can be efficiently found using a primality test.
   - p and q are kept secret.
2. Compute $n = pq$.
   - n is used as the modulus for both the public and private keys. Its length, usually expressed in bits, is the key length.
   - n is released as part of the public key.
3. Compute $\lambda(n)$, where λ is Carmichael's totient function. Since $n = pq$, $\lambda(n) = \operatorname{lcm}(\lambda(p), \lambda(q))$, and since p and q are prime, $\lambda(p) = \varphi(p) = p - 1$, and likewise $\lambda(q) = q - 1$. Hence $\lambda(n) = \operatorname{lcm}(p - 1, q - 1)$.
   - $\lambda(n)$ is kept secret.
   - The lcm may be calculated through the Euclidean algorithm, since $\operatorname{lcm}(a, b) = |ab|/\gcd(a, b)$.
4. Choose an integer e such that $1 < e < \lambda(n)$ and $\gcd(e, \lambda(n)) = 1$; that is, e and $\lambda(n)$ are coprime.
   - e having a short bit-length and small Hamming weight results in more efficient encryption – the most commonly chosen value for e is $2^{16} + 1 = 65\,537$. The smallest (and fastest) possible value for e is 3, but such a small value for e has been shown to be less secure in some settings.[15]
   - e is released as part of the public key.
5. Determine d as $d \equiv e^{-1} \pmod{\lambda(n)}$; that is, d is the modular multiplicative inverse of e modulo $\lambda(n)$.
   - This means: solve for d the equation $d \cdot e \equiv 1 \pmod{\lambda(n)}$; d can be computed efficiently by using the extended Euclidean algorithm, since, thanks to e and $\lambda(n)$ being coprime, said equation is a form of Bézout's identity, where d is one of the coefficients.
   - d is kept secret as the *private key exponent*.

The *public key* consists of the modulus n and the public (or encryption) exponent e. The *private key* consists of the private (or decryption) exponent d, which must be kept secret. p, q, and $\lambda(n)$ must also be kept secret because they can be used to calculate d. In fact, they can all be discarded after d has been computed.[16]

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Prime Numbers

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Modular Arithmetic

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   - The lcm may be calculated through the Euclidean algorithm, since $\operatorname{lcm}(a, b)$

Modular Multiplicative Inverse

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Bezout's Theorem

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Extended Euclidian Algorithm

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Framing Device

We're going to give you enough background to (mostly) understand the RSA encryption system.

Encryption [edit]

After Bob obtains Alice's public key, he can send a message M to Alice.

To do it, he first turns M (strictly speaking, the un-padded plaintext) into an integer m (strictly speaking, the padded plaintext), such that $0 \leq m < n$ by using an agreed-upon reversible protocol known as a padding scheme. He then computes the ciphertext c, using Alice's public key e, corresponding to

$$c \equiv m^e \pmod{n}.$$

This can be done reasonably quickly, even for very large numbers, using modular exponentiation. Bob then transmits c to Alice. Note that at least nine values of m will yield a ciphertext c equal to m,[22] but this is very unlikely to occur in practice.

Decryption [edit]

Alice can recover m from c by using her private key exponent d by computing

$$c^d \equiv (m^e)^d \equiv m \pmod{n}.$$

Given m, she can recover the original message M by reversing the padding scheme.

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Modular Exponentiation

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Divides

Divides

For integers x, y we say $x|y$ ("x divides y") iff there is an integer z such that $xz = y$.

"x is a divisor of y" or "x is a factor of y" means (essentially) the same thing as "x divides y".

("essentially" because of edge cases like when a number is negative or $y = 0$)

"The small number goes first"

Divides

Divides

For integers x, y we say $x|y$ ("x divides y") iff there is an integer z such that $xz = y$.

Which of these are true?

$2|4$ $\quad$ $4|2$ $\quad$ $2|-2$

$5|0$ $\quad$ $0|5$ $\quad$ $1|5$

Divides

Divides

For integers x, y we say $x|y$ ("x divides y") iff there is an integer z such that $xz = y$.

Which of these are true?

$2|4$ True

$4|2$ False

$2|-2$ True

$5|0$ True

$0|5$ False

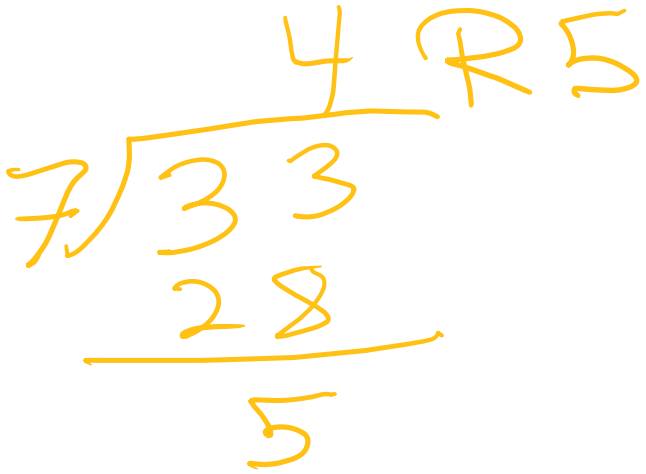
$1|5$ True

A useful theorem

The Division Theorem

For every $a \in \mathbb{Z}$, $\boldsymbol{d} \in \mathbb{Z}$ with $\boldsymbol{d > 0}$
There exist *unique* integers q, r with $0 \leq r < d$
Such that $a = dq + r$

Remember when non integers were still secret, you did division like this?



q is the "quotient"
r is the "remainder"

Unique

The Division Theorem

For every $a \in \mathbb{Z}$, $\boldsymbol{d} \in \mathbb{Z}$ with $\boldsymbol{d > 0}$
There exist *unique* integers q, r with $0 \leq r < d$
Such that $a = dq + r$

"unique" means "only one"....but be careful with how this word is used.

r is unique, **given** a, d. – it still depends on a, d but once you've chosen a and d

"unique" is not saying $\exists r \forall a, d \quad P(a, d, r)$
It's saying $\forall a, d \exists r [P(a, d, r) \wedge [P(a, d, x) \rightarrow x = r]]$

A useful theorem

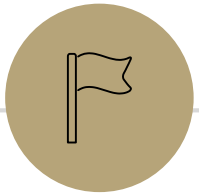
The Division Theorem

For every $a \in \mathbb{Z}$, $\boldsymbol{d} \in \mathbb{Z}$ with $\boldsymbol{d > 0}$
There exist *unique* integers q, r with $0 \leq r < d$
Such that $a = dq + r$

The q is the result of `a/d` (integer division) in Java

The r is the result of `a%d` in Java

That's slightly a lie, r is always non-negative, Java's % operator sometimes gives a negative number.



Extra Set Practice

Extra Set Practice

Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Proof:
Firse, we'll show: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
Let x be an arbitrary element of$A \cup (B \cap C)$.
Then by definition of ∪,∩ we have:
$x \in A \vee (x \in B \wedge x \in C)$
Applying the distributive law, we get
$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
Applying the definition of union, we have:
$x \in (A \cup B)$ and $x \in (A \cup C)$
By definition of intersection we have $x \in (A \cup B) \cap (A \cup C)$.
So $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now we show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
Let x be an arbitrary element of $(A \cup B) \cap (A \cup C)$.
By definition of intersection and union, $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
Applying the distributive law, we have $x \in A \vee (x \ \in B \wedge x \in C)$
Applying the definitions of union and intersection, we have $x \in A \cup (B \cap C)$
So $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
Combining the two directions, since both sets are subsets of each other, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Extra Set Practice

Suppose $A \subseteq B$. Show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Let A, B be arbitrary sets such that $A \subseteq B$.

Let X be an arbitrary element of $\mathcal{P}(A)$.

By definition of powerset, $X \subseteq A$.

Since $X \subseteq A$, every element of X is also in A. And since $A \subseteq B$, we also have that every element of X is also in B.

Thus $X \in \mathcal{P}(B)$ by definition of powerset.

Since an arbitrary element of $\mathcal{P}(A)$ is also in $\mathcal{P}(B)$, we have $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Extra Set Practice

Disprove: If $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$

Consider $A = \{1,2,3\}, B = \{1,2\}, C = \{3,4\}$.

$B \cup C = \{1,2,3,4\}$ so we do have $A \subseteq (B \cup C)$, but $A \nsubseteq B$ and $A \nsubseteq C$.

When you disprove a ∀, you're just providing a counterexample (you're showing ∃) – your proof won't have "let x be an arbitrary element of A."