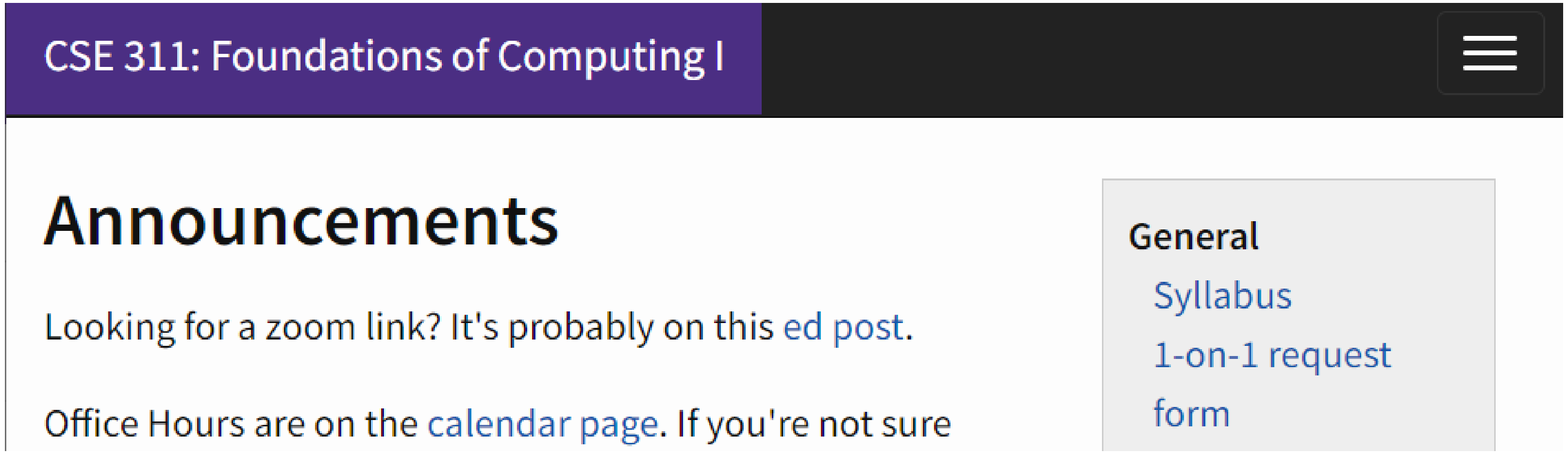
English Proofs and Sets

CSE 311 Winter 23
Lecture 9

Announcements

We have a new form on the homepage for “one-on-one” meetings



It’ll take us a few days to schedule, so not a good option for “I have a question about the current homework” but nice if you have a “topic X from last week never clicked, can we go back?”

Warm-up

Even

An integer x is even if (and only if) there exists an integer z, such that $x = 2z$.

Let your domain of discourse be integers.

Let $\texttt{Even}(x) := \exists y(x = 2y)$.

Prove "if x is even then x^2 is even."

Write a symbolic proof (with the extra rules "Definition of $\texttt{Even}$" and "Algebra").

Then we'll write it in English.

What's the claim in symbolic logic? $\forall x(\texttt{Even}(x) \rightarrow \texttt{Even}(x^2))$

Breakdown the statement

“if x is even then x^2 is even.”

In symbols, that’s: $\forall x\left(\mathbf{Even}(x) \rightarrow \mathbf{Even}\left(x^2\right)\right)$

Let’s break down the statement to understand what the proof needs to look like:

$\forall x$ comes first. We need to introduce an arbitrary variable

$\mathbf{Even}(x) \rightarrow \mathbf{Even}(x^2)$ is left. We prove implications by assuming the hypothesis and setting the conclusion as our goal

$\mathbf{Even}(x)$ is our starting assumption, $\mathbf{Even}(x^2)$ is our goal

If x is even, then x^2 is even.

1. Let a be arbitrary	
2.1 $\texttt{Even}(a)$	Assumption
2.2 ?	?
2.3 ?	?
2.4 ?	?
2.5 ?	?
2.6 ?	?
2.7 $\texttt{Even}(a^2)$	?
3. $\texttt{Even}(a) \rightarrow \texttt{Even}(a^2)$	Direct Proof Rule (2.1-2.7)
4. $\forall x(\texttt{Even}(x) \rightarrow \texttt{Even}(x^2))$	Intro ∀ (3)

If x is even, then x^2 is even.

1. Let a be arbitrary

2.1 $\texttt{Even}(a)$	Assumption
2.2 $\exists y\ (2y = a)$	Definition of Even (2.1)
2.3 $2z = a$	Elim $\exists$(2.2)
2.4 $a^2 = 4z^2$	Algebra (2.3)
2.5 $a^2 = 2 \cdot 2z^2$	Alegbra (2.4)
2.6 $\exists w(2w = a^2)$	Intro $\exists$ (2.5)
2.7 $\texttt{Even}(a^2)$	Definition of Even
3. $\texttt{Even}(a) \rightarrow \texttt{Even}(a^2)$	Direct Proof Rule (2.1-2.7)
4. $\forall x(\texttt{Even}(x) \rightarrow \texttt{Even}(x^2))$	Intro $\forall$ (3)

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Let x be an arbitrary even integer.
By definition, there is an integer y such that $2y = x$.

Squaring both sides, we see that $x^2 = 4y^2 = 2 \cdot 2y^2$.

Because y is an integer, $2y^2$ is also an integer, and x^2 is two times an integer. Thus x^2 is even by the definition of even.
Since x was an arbitrary even integer, we can conclude that for every even x, x^2 is also even.

Converting to English

Start by introducing your assumptions.

Introduce variables with “let.” Introduce assumptions with “suppose.”

Always state what type your variable is. English proofs don’t have an established domain of discourse.

Don’t just use “algebra” explain what’s going on.

We don’t explicitly intro/elim $\exists/\forall$ so we end up with fewer “dummy variables”

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Let's do another!

First a definition

Rational

A real number x is rational if (and only if) there exist integers p and q, with $q \neq 0$ such that $x = p/q$.

$$\texttt{Rational}(x) \coloneqq \exists p \exists q(\texttt{Integer}(p) \land \texttt{Integer}(q) \land (x = p/q) \land q \neq 0)$$

A Word on Definitions

Definitions are critical for CS.

We introduce new concepts, we need to agree on what we mean!

In order to convince me “this number is even”

Most of our proofs boil down to “what are the definitions involved in what I’m showing?” and “how do I verify that definition holds.”

A definition is inherently an if and only if; people sometimes write just “if”; the other direction is “implied” by it being labeled as “the definition.”

Let’s do another!

“The product of two rational numbers is rational.”

What is this statement in predicate logic?

$\forall x \forall y([\texttt{rational}(x) \land \texttt{rational}(y)] \rightarrow \texttt{rational}(xy))$

Remember unquantified variables in English are implicitly universally quantified.

Doing a Proof

$\forall x \forall y([\texttt{rational}(x) \wedge \texttt{rational}(y)] \rightarrow \texttt{rational}(xy))$

"The product of two rational numbers is rational."

DON'T just jump right in!

Look at the statement, make sure you know:

1. What every word in the statement means.
2. What the statement as a whole means.
3. Where to start.
4. What your target is.

Let’s do another!

“The product of two rational numbers is rational.”

Let x, y be arbitrary rational numbers.

Therefore, xy is rational.

Since x and y were arbitrary, we can conclude the product of two rational numbers is rational.

Let’s do another!

“The product of two rational numbers is rational.”

Let x, y be arbitrary rational numbers.

By the definition of rational, $x = a/b$, $y = c/d$ for integers a, b, c, d where $b \neq 0$ and $d \neq 0$.

Multiplying, $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Since integers are closed under multiplication, ac and bd are integers.

Moreover, $bd \neq 0$ because neither b nor d is 0. Thus xy is rational.

Since x and y were arbitrary, we can conclude the product of two rational numbers is rational.

Now You Try

The sum of two even numbers is even.

1. Write the statement in predicate logic.
2. Write an English proof.
3. If you have lots of extra time, try writing the symbolic proof instead.

Now You Try

The sum of two even numbers is even.

Make sure you know:

1. What every word in the statement means.
2. What the statement as a whole means.
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Even

An integer x is even if (and only if) there exists an integer z, such that $x = 2z$.

Pollev.com/robbie

Help me adjust my explanation!

1. Write the statement in predicate logic.
2. Write an English proof.
3. If you have lots of extra time, try writing the symbolic proof instead.

Here's What I got.

$\forall x \forall y([\texttt{Even}(x) \wedge \texttt{Even}(y)] \rightarrow \texttt{Even}(x+y))$

Let x, y be arbitrary integers, and suppose x and y are even.

By the definition of even, $x = 2a, y = 2b$ for some integers a and b.

Summing the equations, $x + y = 2a + 2b = 2(a + b)$.

Since a and b are integers, $a + b$ is an integer, so $x + y$ is even by the definition of even.

Since x, y were arbitrary, we can conclude the sum of two even integers is even.

Why English Proofs?

Those symbolic proofs seemed pretty nice. Computers understand them, and can check them.

So what’s up with these English proofs?

They’re far easier for **people** to understand.

But instead of a computer checking them, now a human is checking them.

Sets

Sets

A set is an **unordered** group of **distinct** elements.

We'll always write a set as a list of its elements inside {curly, brackets}.

Variable names are capital letters, with lower-case letters for elements.

$|A| = 2$. "The size of A is 2." or "A has cardinality 2."

$A = \{\text{curly, brackets}\}$

$B = \{0,5,8,10\} = \{5,0,8,10\} = \{0,0,5,8,10\}$

$C = \{0,1,2,3,4, \dots\}$

Sets

Some more symbols:

$a \in A$ ("a is in A" or "a is an element of A") means a is one of the members of the set.

For $B = \{0,5,8,10\}, \;\; 0 \in B$.

$A \subseteq B$ (A is a subset of B) means every element of A is also in B.

For $A = \{1,2\}$, $B = \{1,2,3\}$ $A \subseteq B$

Sets

Be careful about these two operations:

If $A = \{1,2,3,4,5\}$

$\{1\} \subseteq A$, but $\{1\} \notin A$

$\in$ asks: is this item in that box?

$\subseteq$ asks: is everything in this box also in that box?

Try it!

Let $A = \{1,2,3,4,5\}$

$B = \{1,2,5\}$

Is $A \subseteq A$? Yes!

Is $B \subseteq A$? Yes

Is $A \subseteq B$? No

Is $\{1\} \in A$? No

Is $1 \in A$? Yes

Definitions

$A \subseteq B$ ("A is a subset of B") iff every element of A is also in B.

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$A = B$ ("A equals B") iff A and B have identical elements.

$$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$$

Proof Skeleton

How would we show $A \subseteq B$?

$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

Let x be an arbitrary element of A

…

So x is also in B.

Since x was an arbitrary element of A, we have that $A \subseteq B$.

Proof Skeleton

That wasn't a "new" skeleton! It's exactly what we did last week when we wanted to prove $\forall x(P(x) \to Q(x))$!

What about $A = B$?

$$A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \land B \subseteq A$$

Just do two subset proofs!

i.e. $\forall x(x \in A \to x \in B)$ and $\forall x(x \in B \to x \in A)$

What do we do with sets?

We combined propositions with $\vee, \wedge, \neg$.

We combine sets with $\cap$ [intersection], $\cup$, [union] $\bar{}$[complement]

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$\bar{A} = \{x : x \notin A\}$$

That's a lot of elements...if we take the complement, we'll have some "universe" $\mathcal{U}$, and $\bar{A} = \{x : x \in U \wedge x \notin A\}$
It's a lot like the domain of discourse.