Warm up translate to predicate logic:
"For every x, if x is prime, then x is odd or $x = 2$."

Inference Proofs and Nested Unalike Quantifiers

CSE 311 Winter 23
Lecture 6

Announcements

HW1 grades came back yesterday.

HW1 solutions are at the front, grab one after class. If you don't make it in-person, they will be **outside** my office.

HW3 comes out tonight (available on the webpage).

Today

A lecture in 2 parts.

Part 1: More on quantifiers

What happens when we want to talk about just part of our domain of discourse?

$\forall, \exists$ in the same sentence

Part 2: Inference proofs: how do we prove implications?

Quantifiers

Quantifiers

∀ (for **A**ll) and ∃ (there **E**xists)

Write these statements in predicate logic with quantifiers. Let your domain of discourse be “cats”

This sentence implicitly makes a statement about all cats!

If a cat is fat, then it is happy.

$$\forall x[\texttt{Fat}(x) \rightarrow \texttt{Happy}(x)]$$

Quantifiers

Writing implications can be tricky when we change the domain of discourse.

For every cat: if the cat is fat, then it is happy.

Domain of Discourse: cats $\forall x[\texttt{Fat}(x) \to \texttt{Happy}(x)]$

What if we change our domain of discourse to be all mammals?

We need to limit x to be a cat. How do we do that?

$\forall x[(\texttt{Cat}(x) \land \texttt{Fat}(x)) \to \texttt{Happy}(x)]$ $\forall x[\texttt{Cat}(x) \land (\texttt{Fat}(x) \to \texttt{Happy}(x))]$

Quantifiers

Which of these translates "For every cat: if a cat is fat then it is happy."
when our domain of discourse is "mammals"?

$\forall x[(\mathtt{Cat}(x) \land \mathtt{Fat}(x)) \to \mathtt{Happy}(x)]$

For all mammals, if x is a cat and fat then it is happy
[if x is not a cat, the claim is vacuously true, you can't use the promise for anything]

$\forall x[\mathtt{Cat}(x) \land (\mathtt{Fat}(x) \to \mathtt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[what if x is a dog? Dogs are in the domain, but...uh-oh. This isn't what we meant.]

To "limit" variables to a portion of your domain of discourse under a universal quantifier add a hypothesis to an implication.

Quantifiers

Existential quantifiers need a different rule:

To “limit” variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

There is a dog who is not happy.

Domain of discourse: dogs
$\exists x(\neg \texttt{Happy}(x))$

Quantifiers

Which of these translates “There is a dog who is not happy.”
when our domain of discourse is “mammals”?

$\exists x[\mathtt{Dog}(x) \to \neg\mathtt{Happy}(x))]$

There is a mammal, such that if x is a dog then it is not happy.
[this can’t be right – plug in a cat for x and the implication is true]

$\exists x[(\mathtt{Dog}(x) \land \neg\mathtt{Happy}(x))]$

There is a mammal that is both a dog and not happy.
[this one is correct!]

To “limit” variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

Negating Quantifiers

What happens when we negate an expression with quantifiers?

What does your intuition say?

Original	Negation
Every positive integer is prime	There is a positive integer that is not prime.
$\forall x\ \texttt{Prime}(x)$ Domain of discourse: positive integers	$\exists x(\neg\ \texttt{Prime}(x))$ Domain of discourse: positive integers

Negating Quantifiers

Let's try on an existential quantifier...

Original	Negation
There is a positive integer which is prime and even.	Every positive integer is composite or odd.
$\exists x(\texttt{Prime}(x) \land \texttt{Even}(x))$ Domain of discourse: positive integers	$\forall x(\neg\texttt{Prime}(x) \lor \neg\texttt{Even}(x))$ Domain of discourse: positive integers

To negate an expression with a quantifier

1. Switch the quantifier ($\forall$ becomes $\exists$, $\exists$ becomes $\forall$)
2. Negate the expression inside

Negation

Translate these sentences to predicate logic, then negate them.

All cats have nine lives.

$$\forall x\big(Cat(x) \rightarrow NumLives(x, 9)\big)$$

$\exists x(Cat(x) \wedge \neg(NumLives(x, 9)))$ "There is a cat without 9 lives.

All dogs love every person.

$$\forall x \forall y\big(Dog(x) \wedge Human(y) \rightarrow Love(x, y)\big)$$

$\exists x \exists y(Dog(x) \wedge Human(y) \wedge \neg Love(x, y))$ "There is a dog who does not love someone." "There is a dog and a person such that the dog doesn't love that person."

There is a cat that loves someone.

$$\exists x \exists y(Cat(x) \wedge Human(y) \wedge Love(x, y)$$

$$\forall x \forall y(Cat(x) \wedge Human(y) \rightarrow \neg Love(x, y))$$

"For every cat and every human, the cat does not love that human."

"Every cat does not love any human" ("no cat loves any human")

Negation with Domain Restriction

$\exists x \exists y(Cat(x) \land Human(y) \land Love(x, y))$

$\forall x \forall y([Cat(x) \land Human(y)] \rightarrow \neg Love(x, y))$

There are lots of equivalent expressions to the second. This one is by far the best because it reflects the domain restriction happening. How did we get there?

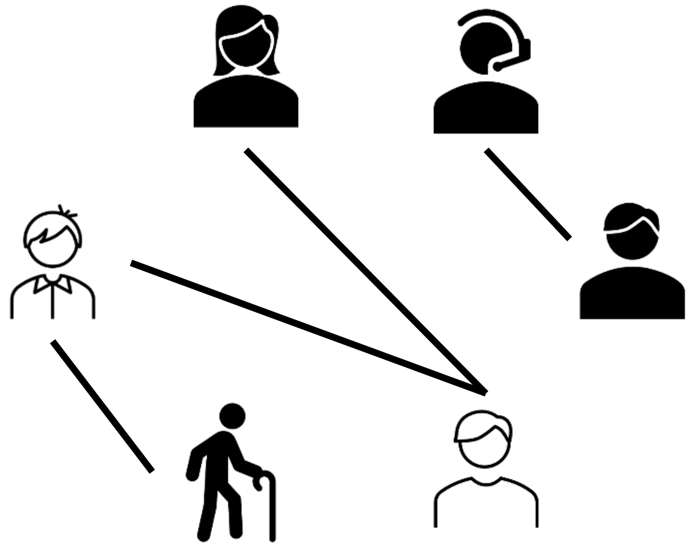
There's a problem in this week's section handout showing similar algebra.

Nested Quantifiers

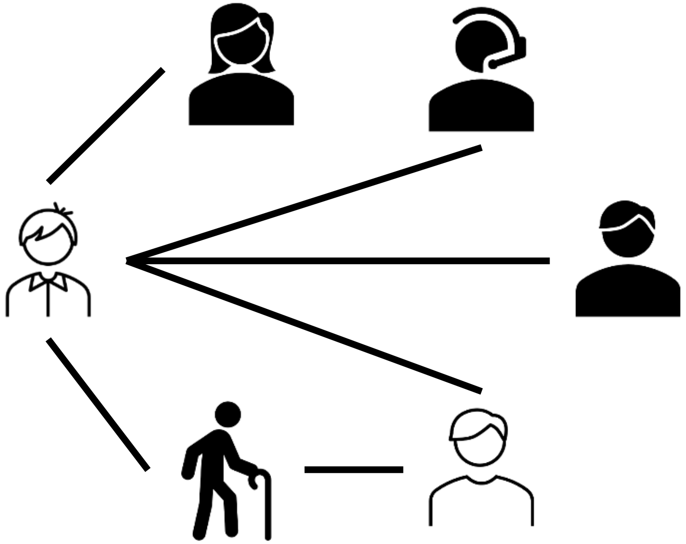
Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x, y)

Everyone is friends with someone.



Someone is friends with everyone.



Nested Quantifiers

Translate these sentences using only quantifiers and the predicate $\texttt{AreFriends}(x, y)$

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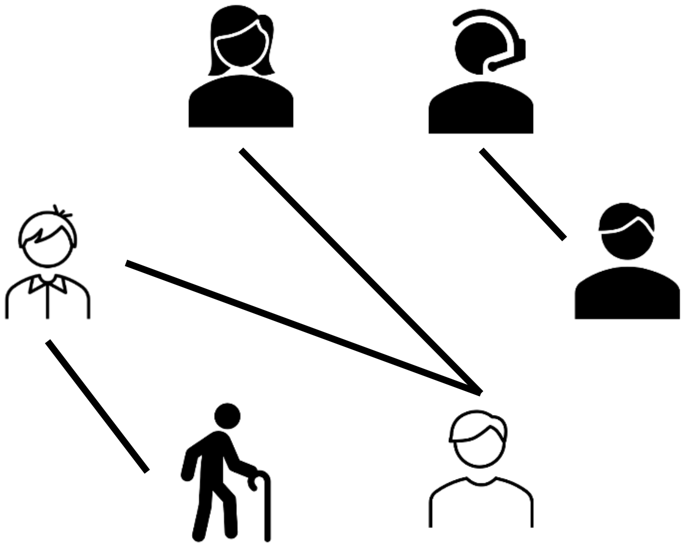

$\forall x (\exists y\ \texttt{AreFriends}(x, y))$

$\forall x \exists y\ \texttt{AreFriends}(x, y)$

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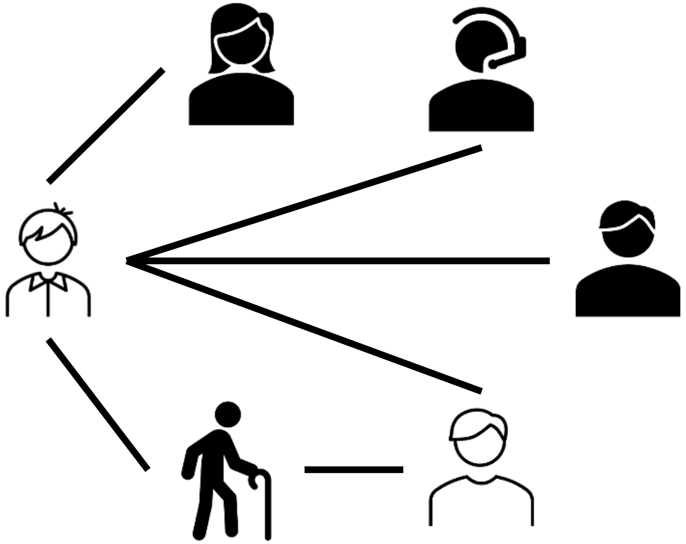


$\exists x (\forall y\ \texttt{AreFriends}(x, y))$

$\exists x \forall y\ \texttt{AreFriends}(x, y)$

Nested Quantifiers

$\forall x \exists y\, P(x, y)$

"For every x there exists a y such that $P(x, y)$ is true."

y might change depending on the x (people have different friends!).

$\exists x \forall y\, P(x, y)$

"There is an x such that for all y, $P(x, y)$ is true."

There's a special, magical x value so that $P(x, y)$ is true regardless of y.

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

$\forall x \exists y P(x, y)$

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

A row, where every entry is $\mathtt{T}$

$\forall x \exists y P(x, y)$

In every row there must be a $\mathtt{T}$

$P(x, y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Keep everything in order

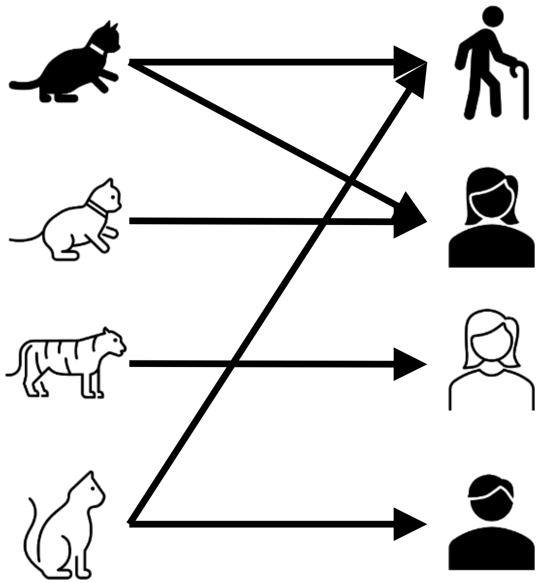
Keep the quantifiers in the same order in English as they are in the logical notation.

"There is someone out there for everyone" is a $\forall x \exists y$ statement in "everyday" English.

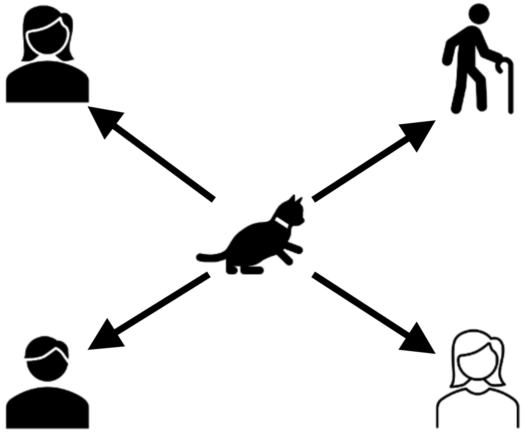
It would **never** be phrased that way in "mathematical English" We'll only every write "for every person, there is someone out there for them."

Try it yourselves

Every cat loves some human.



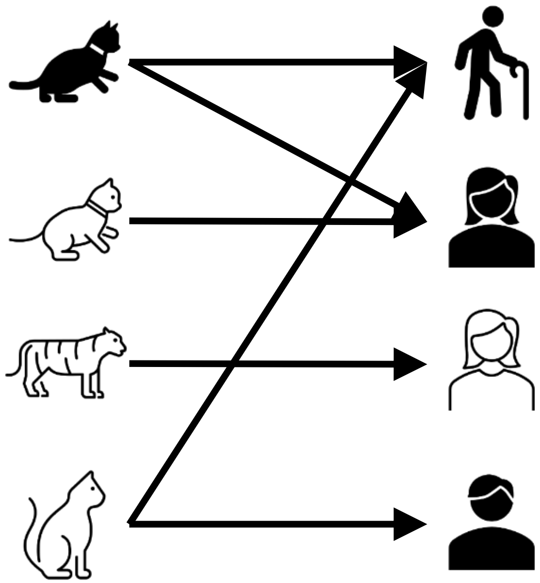
There is a cat that loves every human.



Let your domain of discourse be mammals.
Use the predicates $\texttt{Cat}(x)$, $\texttt{Dog}(x)$, and $\texttt{Loves}(x, y)$ to mean x loves y.

Try it yourselves

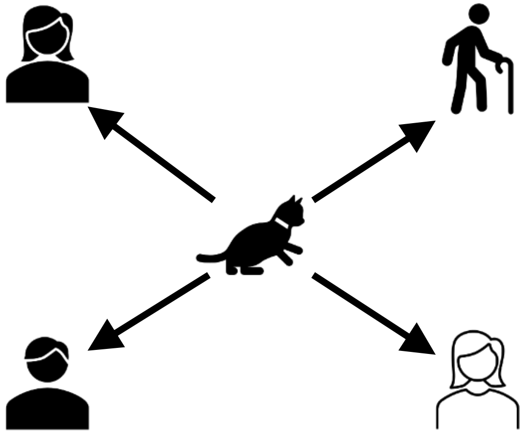
Every cat loves some human.



$\forall x\, (\mathtt{Cat}(x) \rightarrow \exists y[\mathtt{Human}(y) \wedge \mathtt{Loves}(x,y)])$

$\forall x \exists y (\mathtt{Cat}(x) \rightarrow [\mathtt{Human}(y) \wedge \mathtt{Loves}(x,y)])$

There is a cat that loves every human.



$\exists x\, (\mathtt{Cat}(x) \wedge \forall y[\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

$\exists x \forall y (\mathtt{Cat}(x) \wedge [\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

Negation

How do we negate nested quantifiers?

The old rule still applies.

To negate an expression with a quantifier
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

$$\neg(\forall x \exists y \forall z\, [P(x,y) \land Q(y,z)])$$

$$\exists x(\neg(\exists y \forall z\, [P(x,y) \land Q(y,z)]))$$

$$\exists x \forall y(\neg(\forall z[P(x,y) \land Q(y,z)]))$$

$$\exists x \forall y \exists z(\neg[P(x,y) \land Q(y,z)])$$

$$\exists x \forall y \exists z[\neg P(x,y) \lor \neg Q(y,z)]$$

More Translation

For each of the following, translate it, then say whether the statement is true. Let your domain of discourse be integers.

For every integer, there is a greater integer.

$\forall x \exists y(\texttt{Greater}(y, x))$ (This statement is true: y can be $x + 1$ [y depends on x])

There is an integer x, such that for all integers y, xy is equal to 1.

$\exists x \forall y(\texttt{Equal}(xy, 1))$ (This statement is false: no single value of x can play that role for every y.)

$\forall y \exists x(\texttt{Equal}(x + y, 1))$

For every integer, y, there is an integer x such that $x + y = 1$
(This statement is true, y can depend on x)

More Inference Proofs

Try it yourselves

Suppose you know $p \rightarrow q, \neg s \rightarrow \neg q$,and p.
Give an argument to conclude s.

1.	$p \rightarrow q$	Given
2.	$\neg s \rightarrow \neg q$	Given
3.	p	Given
4.	q	Modus Ponens 1,3
5.	$q \rightarrow s$	Contrapositive of 2
6.	s	Modus Ponens 5,4

More Inference Rules

In total, we have two for ∧ and two for ∨, one to create the connector, and one to remove it.

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Intro ∧

$$\frac{A; B}{\therefore \quad A \wedge B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

None of these rules are surprising, but they are useful.

The Direct Proof Rule

We've been implicitly using another "rule" today, the direct proof rule

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \rightarrow B}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
It's an observation that we've done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule…starting right now.

How would you argue…

Let’s say you have a piece of code.

And you think **if** the code gets null input **then** a nullPointerExecption will be thrown.

How would you convince your friend?

You’d probably trace the code, assuming you would get null input.

The code was your **given**

The null input is an assumption

In general

How do you convince someone that $p \rightarrow q$ is true given some surrounding context/some surrounding givens?

You suppose p is true (you assume p)

And then you'll show q must also be true.

Just from p and the Given information.

The Direct Proof Rule

$$\frac{\text{Write a proof ``assume } A \text{ conclude } B\text{''}}{A \to B}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \to B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
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Given: $((p \to q) \land (q \to r))$
Show: $(p \to r)$

Here's an incorrect proof.

1.	$(p \to q) \land (q \to r)$	Given
2.	$p \to q$	Eliminate $\land$ (1)
3.	$q \to r$	Eliminate $\land$ (1)
4.	p	Given???
5.	q	Modus Ponens 4,2
6.	r	Modus Ponens 5,3
7.	$p \to r$	Direct Proof Rule

Given: $((p \rightarrow q) \wedge (q \rightarrow r))$
Show: $(p \rightarrow r)$

Here's an incorrect proof.

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts...

1.	$(p \rightarrow q) \wedge (q \rightarrow r)$	
2.	$p \rightarrow q$	
3.	$q \rightarrow r$	Eliminate $\wedge$ (1)
4.	p	Given ????
5.	q	Modus Ponens 4,2
6.	r	Modus Ponens 5,3
7.	$p \rightarrow r$	Direct Proof Rule

These facts depend on p.
But p isn't known generally.
It was assumed for the purpose of proving $p \rightarrow r$.

Given: $((p \to q) \wedge (q \to r))$
Show: $(p \to r)$

Here's an incorrect proof.

Proofs are supposed to be lists of facts.
Some of these "facts" aren't really facts…

1. $(p \to q) \wedge (q \to r)$
2. $p \to q$
3. $q \to r$ Eliminate $\wedge$ (1)
4. p Given ????
5. q Modus Ponens 4,2
6. r Modus Ponens 5,3
7. $p \to r$ Direct Proof Rule

These facts depend on p.
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Given: $((p \rightarrow q) \wedge (q \rightarrow r))$
Show: $(p \rightarrow r)$

Here's a corrected version of the proof.

1. $(p \rightarrow q) \wedge (q \rightarrow r)$ Given
2. $p \rightarrow q$ Eliminate ∧ 1
3. $q \rightarrow r$ Eliminate ∧ 1
   4.1 p Assumption
   4.2 q Modus Ponens 4.1,2
   4.3 r Modus Ponens 4.2,3
5. $p \rightarrow r$ Direct Proof Rule

When introducing an assumption to prove an implication: Indent, and change numbering.

When reached your conclusion, use the Direct Proof Rule to observe the implication is a fact.

The conclusion is an unconditional fact (doesn't depend on p) so it goes back up a level

Try it!

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

Eliminate ∧
$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨
$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧
$$\frac{A; B}{\therefore \quad A \wedge B}$$

Intro ∨
$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule
$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens
$$\frac{P \rightarrow Q; P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!

Try it!

Given: $p \vee q$, $(r \wedge s) \rightarrow \neg q$, r.
Show: $s \rightarrow p$

1.	$p \vee q$	Given
2.	$(r \wedge s) \rightarrow \neg q$	Given
3.	r	Given
4.1	s	Assumption
4.2	$r \wedge s$	Intro $\wedge$ (3,4.1)
4.3	$\neg q$	Modus Ponens (2, 4.2)
4.4	$q \vee p$	Commutativity (1)
4.5	p	Eliminate $\vee$ (4.4, 4.3)
5.	$s \rightarrow p$	Direct Proof Rule

Inference Rules

Eliminate ∧
$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨
$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧
$$\frac{A; B}{\therefore \quad A \wedge B}$$

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$$\frac{A}{\therefore A \vee B, B \vee A}$$

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