Predicates and Quantifiers

CSE 311 Winter 23
Lecture 5

Announcements

Office hours are shifted next week (Monday is a holiday).

No lecture on Monday, either.

Predicate Logic

So far our propositions have worked great for fixed objects.

What if we want to say "If $x > 10$ then $x^2 > 100$."

$x > 10$ isn't a proposition. Its truth value depends on x.

We need a function that can take in a value for x and output True or False as appropriate.

Predicates

Predicate

A function that outputs true or false.

`Cat(x)` := "`x` is a cat"

`Prime(x)` := "`x` is prime"

`LessThan(x,y)` := "`x<y`"

`Sum(x,y,z)` := "`x+y=z`"

`HasNChars(s,n)` := "string `s` has length `n`"

Numbers and types of inputs can change. Only requirement is output is Boolean.

Analogy

Propositions were like Boolean variables.

What are predicates? Functions that return Booleans

```
public boolean predicate(…)
```

Translation

Translation works a lot like when we just had propositions.

Let's try it…

x is prime or x^2 is odd or $x = 2$.

$\texttt{Prime}(x) \vee \texttt{Odd}(x^2) \vee \texttt{Equals}(x, 2)$

Domain of Discourse

x is prime or x^2 is odd or $x = 2$.

$\texttt{Prime}(x) \vee \texttt{Odd}(x^2) \vee \texttt{Equals}(x, 2)$

Can x be 4.5? What about "abc" ?

I never intended you to plug 4.5 or "abc" into x.

When you read the sentence you probably didn't imagine plugging those values in….

Domain of Discourse

x is prime or x^2 is odd or $x = 2$.

$\texttt{Prime}(x) \vee \texttt{Odd}(x^2) \vee \texttt{Equals}(x, 2)$

To make sure we **can't** plug in 4.5 for x, predicate logic requires deciding on the types we'll allow

Domain of Discourse

The *types* of inputs allowed in our predicates.

Try it…

What's a possible domain of discourse for these lists of predicates?

1. "x is a cat", "x barks", "x likes to take walks"

2. "x is prime", "x=5" "$x < 20$" "x is a power of two"

3. "x is enrolled in course y", "y is a pre-req for z"

Try it…

What's a possible domain of discourse for these lists of predicates?

1. "x is a cat", "x barks", "x likes to take walks"

   "Mammals", "pets", "dogs and cats", …

2. "x is prime", "x=5" "$x < 20$" "x is a power of two"

   "positive integers", "integers", "numbers", …

3. "x is enrolled in course y", "y is a pre-req for z"

   "objects in the university course enrollment system", "university entities", "students and courses", …

More than one domain of discourse might be reasonable…if it might affect the meaning of the statement, we specify it.

Quantifiers

Now that we have variables, let's really use them…

We tend to use variables for two reasons:

1. The statement is true for every x, we just want to put a name on it.
2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

Quantifiers

We have two extra symbols to indicate which way we're using the variable.

1. The statement is true for every x, we just want to put a name on it.

$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, such that $p(x)$ and $q(x)$ are both true.

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Universal Quantifier

"$\forall x$"

"for each x", "for every x", "for all x" are common translations

Remember: upside-down-A for All.

Quantifiers

Existential Quantifier

"$\exists x$"

"there is an x", "there exists an x", "for some x" are common translations

Remember: backwards-E for Exists.

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, for which $p(x)$ and $q(x)$ are both true.

Translations

"For every x, if x is even, then $x = 2$."

"There are x, y such that $\mathrm{x} < y$."

$\exists x\ (\mathtt{Odd}(x) \wedge \mathtt{LessThan}(x, 5))$

$\forall y\ (\mathrm{Even}(y) \wedge \mathrm{Odd}(y))$

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Help me adjust my explanation!

Translations

"For every x, if x is even, then $x = 2$."

$\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

"There are x, y such that x $< y$."

$\exists x \exists y(\texttt{LessThan}(x, y))$

$\exists x\ (\texttt{Odd}(x) \wedge \texttt{LessThan}(x, 5))$

There is an odd number that is less than 5.

$\forall y\ (\mathrm{Even}(y) \wedge \mathrm{Odd}(y))$

All numbers are both even and odd.

Translations

More practice in section and on homework.

Also a reading on the webpage –

An explanation of why "for any" is not a great way to translate ∀ (even though it looks like a good option on the surface)

More information on what happens with multiple quantifiers (we'll discuss more on Wednesday).

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

Evaluating Predicate Logic

"For every x, if x is even, then $x = 2$." / $\forall x(\texttt{Even}(x) \rightarrow \texttt{Equal}(x, 2))$

Is this true?

TRICK QUESTION! It depends on the domain.

Prime Numbers	Positive Integers	Odd integers
True	False	True (vacuously)

One Technical Matter

How do we parse sentences with quantifiers?

What's the "order of operations?"

We will usually put parentheses right after the quantifier and variable to make it clear what's included. If we don't, it's the rest of the expression.

Be careful with repeated variables…they don't always mean what you think they mean.

$\forall x\big(P(x)\big) \wedge \forall x(Q(x))$ are different x's.

Bound Variables

What happens if we repeat a variable?

Whenever you introduce a new quantifier with an already existing variable, it "takes over" that name until its expression ends.

$$\forall x(P(x) \land \forall x[Q(x)] \land R(x))$$

It's common (albeit somewhat confusing) practice to reuse a variables when it "wouldn't matter".

Never do something like the above: where a single name switches from gold to purple back to gold. Switching from gold to purple only is usually fine...but names are cheap.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

For every fruit, if it is not ripe then it is not tasty.

There is a fruit that is sliced and diced.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

$$\exists x(\texttt{Tasty}(x) \wedge \texttt{Ripe}(x))$$

For every fruit, if it is not ripe then it is not tasty.

$$\forall x(\neg\texttt{Ripe}(x) \rightarrow \neg\texttt{Tasty}(x))$$

There is a fruit that is sliced and diced.

$$\exists x(\texttt{Sliced}(x) \wedge \texttt{Diced}(x))$$

Inference Proofs

Inference Proofs

A new way of thinking of proofs:

Here’s one way to get an iron-clad guarantee:

1. Write down all the facts we know.
2. Combine the things we know to derive new facts.
3. Continue until what we want to show is a fact.

Drawing Conclusions

You know “If it is raining, then I have my umbrella”

And “It is raining”

You should conclude…. I have my umbrella!

For whatever you conclude, convert the statement to propositional logic – will your statement hold for any propositions, or is it specific to raining and umbrellas?

I know $(p \rightarrow q)$ and p, I can conclude q
Or said another way: $[(p \rightarrow q) \wedge p] \rightarrow q$

Modus Ponens

The inference from the last slide is always valid. I.e.

$$[(p \rightarrow q) \land p] \rightarrow q$$

Has only True rows in its truth table (it's a tautology)

Modus Ponens – a formal proof

$[(p \to q) \land p] \to q$	$\equiv [(\neg p \lor q) \land p] \to q$	Law of Implication
	$\equiv [p \land (\neg p \lor q)] \to q$	Commutativity
	$\equiv [(p \land \neg p) \lor (p \land q)] \to q$	Distributivity
	$\equiv [\mathrm{F} \lor (p \land q)] \to q$	Negation
	$\equiv [(p \land q) \lor \mathrm{F}] \to q$	Commutativity
	$\equiv [(p \land q)] \to q$	Identity
	$\equiv [\neg(p \land q)] \lor q$	Law of Implication
	$\equiv [\neg p \lor \neg q] \lor q$	DeMorgan's Law
	$\equiv \neg p \lor [\neg q \lor q]$	Associativity
	$\equiv \neg p \lor [q \lor \neg q]$	Commutativity
	$\equiv \neg p \lor \mathrm{T}$	Negation
	$\equiv \mathrm{T}$	Domination

Modus Ponens

The inference from the last slide is always valid. I.e.

$$[(p \rightarrow q) \land p] \rightarrow q \equiv \mathrm{T}$$

We use that inference A LOT

So often people gave it a name ("Modus Ponens")

So often…we don't have time to repeat that 12 line proof EVERY TIME.

Let's make this another law we can apply in a single step.

- Just like refactoring a method in code.

Notation – Laws of Inference

We’re using the “$\rightarrow$ “ symbol A LOT.

Too much

Some new notation to make our lives easier.

$$\frac{\text{If we know } \textbf{both } A \text{ and } B}{\therefore \quad \text{We can conclude any (or all) of } C, D} \qquad \frac{A, B}{\therefore \quad C, D}$$

“$\therefore$” means “therefore” – I knew A, B therefore I can conclude C, D.

$$\frac{p \rightarrow q, p}{\therefore \quad q}$$

Modus Ponens, i.e. $[(p \rightarrow q) \wedge p] \rightarrow q)$, in our new notation.

Another Proof

Let’s keep going.

I know “If it is raining then I have my umbrella” and “I do not have my umbrella”

I can conclude… It is not raining!

What’s the general form? $[(p \to q) \land \neg q] \to \neg p$

How do you think the proof will go?

If you had to convince a friend of this claim in English, how would you do it?

A proof!

We know $p \to q$ and $\neg q$; we want to conclude $\neg p$.
Let's try to prove it. Our goal is to list facts until our goal becomes a fact.
We'll number our facts, and put a justification for each new one.

A proof!

We know $p \to q$ and $\neg q$; we want to conclude $\neg p$.
Let's try to prove it. Our goal is to list facts until our goal becomes a fact.
We'll number our facts, and put a justification for each new one.

1.	$p \to q$	Given
2.	$\neg q$	Given
3.	$\neg q \to \neg p$	Contrapositive of 1.
4.	$\neg p$	Modus Ponens on 3,2.

Try it yourselves

Suppose you know $p \to q, \neg s \to \neg q$,and p.
Give an argument to conclude s.

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Help me adjust my explanation!

Try it yourselves

Suppose you know $p \to q, \neg s \to \neg q$,and p.
Give an argument to conclude s.

1.	$p \to q$	Given
2.	$\neg s \to \neg q$	Given
3.	p	Given
4.	q	Modus Ponens 1,3
5.	$q \to s$	Contrapositive of 2
6.	s	Modus Ponens 5,4

More Inference Rules

We need a couple more inference rules.

These rules set us up to get facts in exactly the right form to apply the really useful rules.

A lot like commutativity and associativity in the propositional logic rules.

Eliminate ∧	$\dfrac{A \wedge B}{\therefore \quad A, B}$	$\dfrac{\text{I know the fact } A \wedge B}{\therefore \text{ I can conclude } A \text{ is a fact and } B \text{ is a fact } \textbf{separately}.}$

More Inference Rules

In total, we have two for ∧ and two for ∨, one to create the connector, and one to remove it.

Eliminate ∧

$$\frac{A \land B}{\therefore \quad A, B}$$

Intro ∧

$$\frac{A, B}{\therefore \quad A \land B}$$

Eliminate ∨

$$\frac{A \lor B, \neg A}{\therefore \quad B}$$

Intro ∨

$$\frac{A}{\therefore A \lor B, B \lor A}$$

None of these rules are surprising, but they are useful.

The Direct Proof Rule

We've been implicitly using another "rule" today, the direct proof rule

$$\frac{\text{Write a proof "given } A \text{ conclude } B\text{"}}{A \rightarrow B}$$

$$\text{Direct Proof rule}\quad \frac{A \Rightarrow B}{A \rightarrow B}$$

This rule is different from the others – $A \Rightarrow B$ is not a "single fact."
It's an observation that we've done a proof. (i.e. that we showed fact B starting from A.)

We will get a lot of mileage out of this rule…starting next time.

Caution

Be careful! Logical inference rules can only be applied to **entire** facts. They cannot be applied to portions of a statement (the way our propositional rules could). Why not?

Suppose we know $p \rightarrow q, r$. Can we conclude q?

1.	$p \rightarrow q$	Given
2.	r	Given
3.	$(p \vee r) \rightarrow q$	Introduce $\vee$ (1)
4.	$p \vee r$	Introduce $\vee$ (2)
5.	q	Modus Ponens 3,4.

Intro $\vee$

$$\frac{A}{\therefore A \vee B, B \vee A}$$

One more Proof

Show if we know: $p, q, [(p \wedge q) \rightarrow (r \wedge s)], r \rightarrow t$ we can conclude t.

One more Proof

Show if we know: $p, q, [(p \wedge q) \rightarrow (r \wedge s)], r \rightarrow t$ we can conclude t.

1.	p	Given
2.	q	Given
3.	$[(p \wedge q) \rightarrow (r \wedge s)]$	Given
4.	$r \rightarrow t$	Given
5.	$p \wedge q$	Intro $\wedge$ (1,2)
6.	$r \wedge s$	Modus Ponens (3,5)
7.	r	Eliminate $\wedge$ (6)
8.	t	Modus Ponens (4,7)

Inference Rules

Eliminate ∧

$$\frac{A \wedge B}{\therefore \quad A, B}$$

Eliminate ∨

$$\frac{A \vee B, \neg A}{\therefore \quad B}$$

Intro ∧

$$\frac{A, B}{\therefore \quad A \wedge B}$$

Intro ∨

$$\frac{A}{\therefore A \vee B, B \vee A}$$

Direct Proof rule

$$\frac{A \Rightarrow B}{A \rightarrow B}$$

Modus Ponens

$$\frac{P \rightarrow Q, P}{\therefore \quad Q}$$

You can still use all the propositional logic equivalences too!