Homework 4: English Proofs

Due date: Wednesday February 1st at 10 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the grading guidelines on the assignments page for more information on what we're looking for.

In order to assist with the transition from formal proofs to English proofs, we've published a style guide on the website containing some tips. This guide contains references to proof materials that we haven't taught yet, so don't worry if some of these terms are unfamiliar.

1. Sets Concept Check [11 points]

There are lots of definitions and symbols with sets. To make sure you've got them all, we have a few multiple choice questions on gradescope in a "checkpoint." We **strongly** encourage you to do this problem first. It's hard to write a proof about sets if you aren't 100% sure what the symbols are saying!

Gradescope will grade your submission automatically. You can submit as many times as you want until you have everything correct.

2. English Spoofs and Confusion [12 points]

2.1. What's wrong with this picture? [6 points]

Consider the following statement:

For all real numbers a, b, and c, if $ab = bc$, then $a = c$.

And the following spoof (incorrect proof) of the statement:

> Let a and c be arbitrary real numbers such that $a = c$. Let b be an arbitrary real number. Multiplying both sides of the equation by b, we obtain $ab = bc$.

(a) Why is the above proof incorrect?

(b) Is the original statement true of false? If the statement is true, write a correct proof. If it is false, provide a counterexample.

2.2. What's wrong with that picture? [6 points]

Consider the following statement:

For all real numbers a and b, if $a^2 = b^2$, then $a = b$.

And the following spoof (incorrect proof) of the statement:

> Let a and b be arbitrary real numbers such that $a^2 = b^2$. Since $a^2 \geq 0, b^2 \geq 0$, their square root is a real number and positive. Then, applying the square root function to both sides, we conclude $a = \sqrt{a^2} = \sqrt{b^2} = b$.

(a) Why is the above proof incorrect?

(b) Is the original statement true or false? If the statement is true, write a correct proof. If it is false, provide a counterexample.

3. Formal and English [20 points]

In this problem, we'll practice writing both formal inference proofs and English proofs. Let your domain of discourse be integers.

Define $\texttt{Mysterious}(x)$ to be true if and only if $4 \mid (x-3)$.
The "|" symbol is the symbol for "divides." We'll discuss it in lecture soon, but you can do the proof just knowing the symbols for it. For our domain of discourse being integers, We say that $a|b$ if and only if $\exists k(ak = b)$. Note that all of a, k, b must be integers for this definition. Define $\texttt{Odd}(x)$ to be true if and only if $\exists k(x = 2k + 1)$.

(a) Give a predicate definition of $\texttt{Mysterious}(x)$ that uses an $\exists$ quantifier. [2 points]

(b) Show that $\forall x(\texttt{Mysterious}(x) \rightarrow \texttt{Odd}(x))$, using an inference proof. Let the domain of discourse for your proof be integers. You may use the definitions of predicates in the problem (including your answer to part a), as well as "algebra" to complete the proof. [8 points]

(c) Write an English proof to show that if 4 divides $(x-3)$ for an integer x, then x is odd. Recall that English proofs don't have domains of discourse, so you need to define types for your variables. [8 points]

(d) Go through your English proof, for each sentence in it, state which step(s) of your inference proof it most closely corresponds to (it's okay if a few steps overlap or don't correspond to a particular sentence, but this shouldn't happen to a lot of steps.). [2 points]

4. A Subset Proof [7 points]

Let A, B, C be arbitrary sets. Prove that $(A \cap B) \cup (A \cap C) \subseteq A$. Write an English proof (not an infere proof).

Hint: you will need to use the **proof by case** technique, which we will cover on Friday's lecture, in this question.

5. Set Proofs [16 points]

Let A, B, C be arbitrary sets. For each of the following claims: if it is true, give an **English proof**. If it is false, disprove it with an English proof (If you need to disprove the statement, remember that we've seen only one proof technique in class for disproving a $\forall$).

(a) $(B \backslash A) \cap (C \backslash A) = (B \cap C) \backslash A$

(b) $A \backslash (B \backslash C) = (A \backslash B) \backslash C$

6. I've never seen such raw power[sets] [10 points]

Let S and T be arbitrary sets. Your friend is trying to figure out some other way to express $\mathcal{P}(S \cup T)$.

(a) Your friend presented the following spoof for the claim

$$\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T)$$

This spoof has two errors

- An error in proof strategy, such that even if every step of the proof is correct, it's not a proof of the claim.

- A false assertion in the middle of the proof (i.e., a statement that doesn't follow from prior information).

Identify both errors and explain why they cause the proof to be incorrect (2-3 sentences per error). [6 points]

Spoof. We take some arbitrary element $X \in \mathcal{P}(S \cup T)$. We know that $X \subseteq S \cup T$ by the definition of powerset. This means that $X \subseteq S \vee X \subseteq T$ by the definition of set union. From here, we get $X \in \mathcal{P}(S) \vee X \in \mathcal{P}(T)$ from the definition of powerset. Finally, we argue that this means $X \in \mathcal{P}(S) \cup \mathcal{P}(T)$ by the definition of set union. We know that since X was arbitrarily chosen, this property must hold for any X. Thus since every element in the first set can be understood as an element in the second set, the two sets are equal. □

(b) Now your friend thinks they have another expression that may be equivalent to $\mathcal{P}(S \cup T)$. Prove or disprove your friend's claim. If it is true, give an **English proof**. If it is false, disprove it with an English proof (if you need to disprove the statement, remember that we've seen only one proof technique in class for disproving a $\forall$). [4 points]

$$\mathcal{P}(S \cup T) = \mathcal{P}(S) \cup \mathcal{P}(T) \cup \mathcal{P}(S \cap T)$$

7. Cartesian Products [8 points]

In this problem we will write the skeleton for a proof that $(S \cup T) \times V = (S \times V) \cup (T \times V)$.

(a) It's a common error to take an arbitrary element of a domain, combine it with another arbitrary thing and claim you have a combined arbitrary thing. There is no rule of inference that lets us do this. You can only call something arbitrary if it's the first time you're discussing a fresh variable. Let's see an example where things go wrong: Let x and y be arbitrary positive integers. Suppose you combine them into $x + y$. Explain why $x + y$ is **not** an arbitrary positive integer. (1-3 sentences) Remember that arbitrary means "you can plug in any element of the domain." [4 points]

(b) Now let's write the skeleton for an **English proof**! Introduce any variables you need, any assumptions you can make, and what your target is. If your proof has multiple parts to it, do that for every piece. For examples, see Problem 2 of the Section 4 handout. Be very careful with how you introduce your assumptions; you cannot combine two arbitrary things to get an arbitrary thing. [4 points]

8. Feedback [1 point]

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?