CSE 311: Foundations of Computing I

Section 6: Induction, Midterm Review

1. Find the Bug!

Did you know that all dogs are named Dubs? Let's prove it by induction. The key is talking about groups of dogs, where every dog has the same name.

Let $P(i)$ mean "all groups of i dogs have the same name." We prove $\forall n\, P(n)$ by induction on n.

Base Case: $P(1)$ Take an arbitrary group of one dog, all dogs in that group all have the same name (there's only the one, so it has the same name as itself).

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary k.

Inductive Step: Consider an arbitrary group of $k+1$ dogs. Arbitrarily select a dog, D, and remove it from the group. What remains is a group of k dogs. By inductive hypothesis, all k of those dogs have the same name. Add D back to the group, and remove some other dog D'. We have a (different) group of k dogs, so the inductive hypothesis applies again, and every dog in that group also shares the same name. All $k+1$ dogs appeared in at least one of the two groups, and our groups overlapped, so all of our $k+1$ dogs have the same name, as required.

Conclusion: We conclude $P(n)$ holds for all n by the principle of induction.

Recalling that Dubs is a dog, we have that every dog must have the same name as him, so every dog is named Dubs.

This proof cannot be correct (the proposed claim is false). Where is the bug?

2. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

3. Snack Induction

You want to buy snacks for a party you're throwing. However, the local grocery store only sells snacks in packs of 5 and packs of 7.

Prove that you can buy exactly n snacks for all integers $n \geq 24$.

4. Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- Soy(x) is true iff x contains soy milk.
- Whole(x) is true iff x contains whole milk.

- Sugar(x) is true iff x contains sugar
- Decaf(x) is true iff x is not caffeinated.
- Vegan(x) is true iff x is vegan.
- CadeLikes(x) is true iff Cade likes the drink x.
- RobertLikes(x) is true iff Robert likes the drink x.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

(a) Coffee drinks with whole milk are not vegan.

(b) Robert only likes one coffee drink, and that drink is not vegan.

(c) Unless a drink has whole milk and sugar, Robert won't like it.

(d) Every decaf drink that Cade likes has sugar.

5. Midterm Review: Number Theory

(a) Prove that for all integers x and all integers $p > 1$, if $x \equiv_p 1$, then $x^2 \equiv_p 1$. Try to prove this directly by definitions, instead of using modular arithmetic properties!

Hint: Recall that $(x-1)(x+1) = x^2 - 1$.

(b) Now prove that for all integers x and all **prime** integers p, if $x^2 \equiv_p 1$, then $x \equiv_p 1$ or $x \equiv_p -1$.

Hint: You may the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

(c) We showed in part (b) that if x, p are integers and p is prime, then $x^2 \equiv_p 1$, then $x \equiv_p 1$ or $x \equiv_p -1$. Prove that this claim does not always hold for integers $p > 1$ when p is **not prime**.

6. Midterm Review: Set Theory

(a) Prove or disprove: For all sets A, B, C if $A \cap C = B \cap C$ then $A = B$.

(b) Prove or disprove: For all sets A, B, C if $A \cup C = B \cup C$ then $A = B$.

(c) Prove or disprove: For all sets A, B, C if $A \cup C = B \cup C$ and $A \cap C = B \cap C$ then $A = B$.

7. Midterm Review: Induction

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n+1)(2n+1)$.