

CSE 311: Foundations of Computing I

Section 5: Set Theory, Induction Solutions

1. Computing Cardinality

What is the cardinality of each of these sets? If the set has infinitely many elements, say ∞ .

(a) $A = \{1, 2, 3, 2\}$

Solution:

3

(b) $B = \{ \emptyset, \{ \emptyset \}, \{ \emptyset, \emptyset \}, \{ \emptyset, \emptyset, \emptyset \}, \dots \}$

Solution:

$$\begin{aligned} B &= \{ \emptyset, \{ \emptyset \}, \{ \emptyset, \emptyset \}, \{ \emptyset, \emptyset, \emptyset \}, \dots \} \\ &= \{ \emptyset, \{ \emptyset \} \} \end{aligned}$$

So, there are 2 elements in B .

(c) $C = \{ \emptyset \}$

Solution:

1

(d) $D = \{ \{2\}, \{3, 4\}, \emptyset \} \times C$

Solution:

3

(e) $E = \mathcal{P}(D)$

Solution:

8

2. Subsets

Prove each of the following set identities.

(a) For all sets A, B, C , we have $A \setminus B \subseteq A \cup C$.

Solution:

Let A, B, C be arbitrary sets. Let $x \in A \setminus B$ be arbitrary. By definition of set difference, this means that $x \in A$ and $x \notin B$. Since $x \in A$, we have that $x \in A$ or $x \in C$. So by definition of union, $x \in A \cup C$. Since x was arbitrary, this shows $A \setminus B \subseteq A \cup C$. Since A, B, C were arbitrary sets, the claim holds.

(b) For all sets A, B, C , we have $(A \setminus B) \setminus C \subseteq A \setminus C$.

Solution:

Let A, B, C be arbitrary sets. Let $x \in (A \setminus B) \setminus C$ be arbitrary. By definition of set difference, this means that $x \in A \setminus B$ and $x \notin C$. By definition of set difference again, this means $x \in A$ and $x \notin B$. Since $x \in A$ and $x \notin C$, we have by definition of set difference that $x \in A \setminus C$. Since x was arbitrary, this shows $(A \setminus B) \setminus C \subseteq A \setminus C$. Since A, B, C were arbitrary sets, the claim holds.

(c) For all sets A, B, C, D , we have $(A \cap B) \times C \subseteq A \times (C \cup D)$.

Solution:

Let A, B, C, D be arbitrary sets. Let $x \in (A \cap B) \times C$ be arbitrary. Then, by definition of Cartesian product, x must be of the form (y, z) where $y \in A \cap B$ and $z \in C$. Since $y \in A \cap B$, by definition of intersection, $y \in A$ (and $y \in B$). Since $z \in C$, certainly $z \in C$ or $z \in D$. By definition of union, we also have $z \in C \cup D$. Thus, since $y \in A$ and $z \in C \cup D$, by definition of Cartesian product we have $x = (y, z) \in A \times (C \cup D)$. Since x was an arbitrary element of $(A \cap B) \times C$ we have proved that $(A \cap B) \times C \subseteq A \times (C \cup D)$. Since A, B, C, D were arbitrary sets, the claim holds.

3. Set Equality

Prove using a chain of logical equivalences that that for all sets A, B, C , we have: $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Solution:

Let A, B, C be arbitrary sets. Let x be arbitrary. Then:

$$\begin{aligned} x \in A \setminus (B \cup C) &\equiv x \in A \wedge \neg(x \in (B \cup C)) && \text{[Def of Set Difference]} \\ &\equiv x \in A \wedge \neg(x \in B \vee x \in C) && \text{[Def of Union]} \\ &\equiv x \in A \wedge (x \notin B \wedge x \notin C) && \text{[De Morgan]} \\ &\equiv (x \in A \wedge x \in A) \wedge (x \notin B \wedge x \notin C) && \text{[Idempotency]} \\ &\equiv (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) && \text{[Associativity/Commutativity]} \\ &\equiv (x \in (A \setminus B)) \wedge (x \in (A \setminus C)) && \text{[Def of Set Difference]} \\ &\equiv (x \in (A \setminus B) \cap (A \setminus C)) && \text{[Def of Intersection]} \end{aligned}$$

Since x was arbitrary, we have shown that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. Since A, B, C were arbitrary sets, the claim holds.

4. Power Sets

Prove that for all sets A, B : $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Solution:

Let A, B be arbitrary sets. Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$ be arbitrary. Then by definition of union, $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$.

Case 1: $X \in \mathcal{P}(A)$. Then by definition of powerset, $X \subseteq A$. Let $x \in X$ be arbitrary. Then because $x \in X$ and $X \subseteq A$, we have $x \in A$. Then certainly $x \in A$ or $x \in B$. So $x \in A \cup B$ by definition of union. Since x was arbitrary, $X \subseteq A \cup B$. Thus by definition of powerset, $X \in \mathcal{P}(A \cup B)$.

Case 2: $X \in \mathcal{P}(B)$. Then by definition of powerset, $X \subseteq B$. Let $x \in X$ be arbitrary. Then because $x \in X$ and $X \subseteq B$, we have $x \in B$. Then certainly $x \in A$ or $x \in B$. So $x \in A \cup B$ by definition of union. Since x was arbitrary, $X \subseteq A \cup B$. Thus by definition of powerset, $X \in \mathcal{P}(A \cup B)$.

Thus in all cases, $X \in \mathcal{P}(A \cup B)$. Since X was arbitrary, we have shown that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Since A, B were arbitrary sets, the claim holds for all sets A, B .

5. Induction with Equality

- (a) Define the triangle numbers as $\Delta_n = 0 + 1 + 2 + \dots + n$, where $n \in \mathbb{N}$. It is a fact that $\Delta_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \dots + n^3 = \Delta_n^2$$

Solution:

First, note that $\Delta_n = (0 + 1 + 2 + \dots + n)$. So, we are trying to prove

$$(0^3 + 1^3 + \dots + n^3) = (0 + 1 + \dots + n)^2$$

Proof: Let $P(n)$ be the statement:

$$0^3 + 1^3 + \dots + n^3 = (0 + 1 + \dots + n)^2.$$

We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n .

Base Case. $0^3 = 0^2$, so $P(0)$ holds.

Inductive Hypothesis. Suppose that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Inductive Step. We show $P(k+1)$:

$$\begin{aligned} 0^3 + 1^3 + \dots + (k+1)^3 &= (0^3 + 1^3 + \dots + k^3) + (k+1)^3 && \text{[Associativity]} \\ &= (0 + 1 + \dots + k)^2 + (k+1)^3 && \text{[Inductive Hypothesis]} \\ &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[Proved in class]} \\ &= (k+1)^2 \left(\frac{k^2}{2^2} + (k+1)\right) && \text{[Factor } (k+1)^2\text{]} \\ &= (k+1)^2 \left(\frac{k^2 + 4k + 4}{4}\right) && \text{[Add via common denominator]} \\ &= (k+1)^2 \left(\frac{(k+2)^2}{4}\right) && \text{[Factor numerator]} \\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \text{[Take out the square]} \\ &= (0 + 1 + \dots + (k+1))^2 && \text{[Formula from class again]} \end{aligned}$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of induction.

- (b) For every $n \in \mathbb{N}$, define S_n to be the sum of the squares of the natural numbers up to n , or

$$S_n = 0^2 + 1^2 + \dots + n^2.$$

For all $n \in \mathbb{N}$, prove that $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Solution:

Let $P(n)$ be the statement " $S_n = \frac{1}{6}n(n+1)(2n+1)$ " defined for all $n \in \mathbb{N}$. We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n .

Base Case. $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that $P(0)$ is true.

Induction Hypothesis. Suppose that $P(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Induction Step. Examining S_{k+1} , we see that

$$S_{k+1} = 0^2 + 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the induction hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned} S_{k+1} &= S_k + (k+1)^2 && \text{by definition} \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 && \text{by the I.H.} \\ &= (k+1) \left(\frac{1}{6}k(2k+1) + (k+1) \right) && \text{using common factor } (k+1) \\ &= \frac{1}{6}(k+1)(k(2k+1) + 6(k+1)) \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) && \text{factoring the quadratic term} \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Thus, we can conclude that $P(k+1)$ is true.

Note: We used the fact that we needed to prove $P(k+1)$ as a clue to help us figure out what the factors of that quadratic term might be. If $P(k+1)$ had not been correct then this wouldn't have worked out.

Conclusion: Therefore, $P(n)$ is true for all $n \in \mathbb{N}$ by induction.

6. Induction with Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be " $9 \mid n^3 + (n+1)^3 + (n+2)^3$ ". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case ($n = 2$): $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Induction Hypothesis: Assume that $9 \mid j^3 + (j+1)^3 + (j+2)^3$ for an arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer k by the definition of divides.

Induction Step: Goal: Show $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$

$$\begin{aligned} (j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + 9k - j^3 && \text{[Induction Hypothesis]} \\ &= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\ &= 9j^2 + 27j + 27 + 9k \\ &= 9(j^2 + 3j + 3 + k) \end{aligned}$$

Since j is an integer, $j^2 + 3j + 3 + k$ is also an integer. Therefore, by the definition of divides, $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \rightarrow P(j+1)$ for an arbitrary integer $j > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

7. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Solution:

Let $P(n)$ be “ $6n + 6 < 2^n$ ”. We will prove $P(n)$ for all integers $n \geq 6$ by induction on n .

Base Case ($n = 6$): $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Inductive Hypothesis: Assume that $6k + 6 < 2^k$ for an arbitrary integer $k \geq 6$.

Inductive Step: Goal: Show $6(k + 1) + 6 < 2^{k+1}$

$$\begin{aligned} 6(k + 1) + 6 &= 6k + 6 + 6 \\ &< 2^k + 6 && \text{[Inductive Hypothesis]} \\ &< 2^k + 2^k && \text{[Since } 2^k > 6, \text{ since } k \geq 6\text{]} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

So $P(k) \rightarrow P(k + 1)$ for an arbitrary integer $k \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by the principle of induction.