CSE 311: Foundations of Computing I

Section 5: Set Theory, Induction

1. Computing Cardinality

What is the cardinality of each of these sets? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\ \varnothing,\ \{\ \varnothing\},\ \{\ \varnothing,\ \varnothing\},\ \{\ \varnothing, \varnothing, \varnothing\},\ \dots\}$

(c) $C = \{\varnothing\}$

(d) $D = \{\{2\}, \{3, 4\}, \varnothing\} \times C$

(e) $E = \mathcal{P}(D)$

2. Subsets

Prove each of the following set identities.

(a) For all sets A, B, C, we have $A \setminus B \subseteq A \cup C$.

(b) For all sets A, B, C, we have $(A \setminus B) \setminus C \subseteq A \setminus C$.

(c) For all sets A, B, C, D, we have $(A \cap B) \times C \subseteq A \times (C \cup D)$.

3. Set Equality

Prove using a chain of logical equivalences that that for all sets A, B, C, we have: $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

4. Power Sets

Prove that for all sets A, B: $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

5. Induction with Equality

(a) Define the triangle numbers as $\triangle_n = 0 + 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. It is a fact that $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

(b) For every $n \in \mathbb{N}$, define S_n to be the sum of the squares of the natural numbers up to n, or

$$S_n = 0^2 + 1^2 + \cdots n^2.$$

For all $n \in \mathbb{N}$, prove that $S_n = \frac{1}{6}n(n+1)(2n+1)$.

6. Induction with Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

7. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.