

CSE 311: Foundations of Computing I

Section 3: Proof Techniques Solutions

1. Direct Proof

- (a) Let the domain of discourse be integers. Define the predicates $\text{Odd}(x) := \exists k(x = 2k + 1)$, and $\text{Even}(x) := \exists k(x = 2k)$. Translate the following claim to predicate logic:

The sum of an even and odd integer is odd.

Solution:

$$\forall n \forall m ((\text{Even}(n) \wedge \text{Odd}(m)) \rightarrow \text{Odd}(n + m))$$

- (b) Prove that the claim holds.

Solution:

Let n and m be arbitrary integers. Suppose n is even and m is odd. Then by definition of even, $n = 2k$ for some integer k . By definition of odd, $m = 2j + 1$ for some integer j . Then consider $n + m$:

$$n + m = 2k + 2j + 1 = 2(k + j) + 1$$

Since k and j are integers, $k + j$ is an integer.

Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd. Since n and m were arbitrary, the sum of any even and odd integer is odd.

2. Proof of Biconditional

- (a) Let the domain of discourse be integers. Define the predicates $\text{Odd}(x) := \exists k(x = 2k + 1)$, and $\text{Even}(x) := \exists k(x = 2k)$. Translate the following claim to predicate logic:

For all integers n , $n - 4$ is even if and only if $n + 17$ is odd.

Solution:

$$\forall n (\text{Even}(n - 4) \leftrightarrow \text{Odd}(n + 17))$$

- (b) Prove that the claim holds.

Solution:

\Rightarrow Let n be an arbitrary integer. Suppose that $n - 4$ is even. Then by definition of even, $n - 4 = 2k$ for some integer k . Then observe that:

$$n - 4 = 2k$$

$$n + 17 = 2k + 21$$

$$n + 17 = 2(k + 10) + 1$$

Adding 21 to both sides

Factoring

Thus $n + 17 = 2(k + 10) + 1$. Since k is an integer, $k + 10$ is an integer. So $n + 17$ is 2 times an integer plus 1. Thus by definition of odd, $n + 17$ is odd. Since n was arbitrary, we have shown that for all integers

n that if $n - 4$ is even, then $n + 17$ is odd.

\Leftarrow Let n be an arbitrary integer. Suppose $n + 17$ is odd. Then by definition of odd, $n + 17 = 2k + 1$ for some integer k . Then observe that:

$$n + 17 = 2k + 1$$

$$n - 4 = 2k + 1 - 21$$

Subtracting 21 from both sides

$$n - 4 = 2(k - 10)$$

Factoring

Thus $n - 4 = 2(k - 10)$. Since k is an integer, $k - 10$ is an integer. So $n - 4$ is 2 times an integer. So by definition of even, $n - 4$ is even. Since n was arbitrary, we have shown that for all integers n , if $n + 17$ is odd, then $n - 4$ is even.

3. Proof by Contrapositive

- (a) Let the domain of discourse be integers. Define the predicates $\text{Odd}(x) := \exists k(x = 2k + 1)$, and $\text{Even}(x) := \exists k(x = 2k)$. Translate the following claim to predicate logic:

For all integers x , if $7x + 9$ is even, then x is odd.

Solution:

$$\forall x(\text{Even}(7x + 9) \rightarrow \text{Odd}(x))$$

- (b) Try to prove the claim directly. Do you get stuck?

Note that it is actually possible to write a direct proof, though it is slightly more difficult to see how.

Solution:

Let x be an arbitrary integer. Suppose that $7x + 9$ is even. Then by definition of even, there exists some integer k such that $7x + 9 = 2k$. Then subtracting $6x + 9$ from both sides, we have:

$$x = 2k - 6x - 9 = 2(k - 3x - 5) + 1$$

Since k and x are integers, $k - 3x - 5$ is an integer. So x is 2 times an integer plus 1. So by definition of odd, x is odd. Since x was arbitrary, this shows that for all integers x , if $7x + 9$ is even then x is odd.

- (c) What is the contrapositive of the claim in predicate logic?

Solution:

$$\forall x(\text{Even}(x) \rightarrow \text{Odd}(7x + 9))$$

- (d) Prove that the claim holds by proving the contrapositive.

Solution:

Let x be an arbitrary integer. Suppose that x is even. Then by definition of even, there exists some integer k such that $x = 2k$. Then consider $7x + 9$:

$$7x + 9 = 7(2k) + 9 = 14k + 9 = 2(7k + 4) + 1$$

Since k is an integer, $7k + 4$ is an integer. So $7x + 9$ is 2 times an integer plus 1. So by definition of odd, $7x + 9$ is odd. Since x was arbitrary, this shows that for all integers x , if x is even then $7x + 9$ is odd. Thus the contrapositive also holds: for all integers x , if $7x + 9$ is even, then x is odd.

4. Proof by Cases

Prove by cases that for all integers n , $n^2 - 3n$ is even.

Solution:

Let n be an arbitrary integer.

Case 1: n is even.

Then by definition of even, $n = 2k$ for some integer k . Then consider $n^2 - 3n$:

$$n^2 - 3n = (2k)^2 - 3(2k) = 4k^2 - 6k = 2(2k^2 - 3k)$$

Since k is an integer, $2k^2 - 3k$ is an integer. Then $n^2 - 3n$ is 2 times an integer. So by definition of even, $n^2 - 3n$ is even. Thus we have shown that in this case, $n^2 - 3n$ is even.

Case 2: n is odd.

Then by definition of odd, $n = 2k + 1$ for some integer k . Then consider $n^2 - 3n$:

$$n^2 - 3n = (2k + 1)^2 - 3(2k + 1) = (4k^2 + 4k + 1) - (6k + 3) = 4k^2 - 2k - 2 = 2(2k^2 - k - 1)$$

Since k is an integer, $2k^2 - k - 1$ is an integer. Then $n^2 - 3n$ is 2 times an integer. So by definition of even, $n^2 - 3n$ is even. Thus we have shown that in this case, $n^2 - 3n$ is even.

Conclusion: Thus we have shown that in all cases, $n^2 - 3n$ is even. Since n was arbitrary, for all integers n , $n^2 - 3n$ is even.

5. Disproving a For All Claim

Disprove the following claim:

For all integers a, b, c if $ac = bc$ then $a = b$.

Solution:

Consider $a = 1, b = 2, c = 0$. Then $ac = 0 = bc$ but $a \neq b$. This is a counterexample to the claim.

6. Disproving an Exists Claim

Consider the following claim:

There exists an integer x such that x is even and x^2 is odd.

- (a) This claim is false. Without using any formal reasoning, what does your intuition say about how to disprove this claim?

Solution:

Show that if an integer x is even, then x^2 is also even.

- (b) Let the domain of discourse be integers. Define the predicates $\text{Odd}(x) := \exists k(x = 2k + 1)$, and $\text{Even}(x) := \exists k(x = 2k)$. Translate the above claim to predicate logic.

Solution:

$\exists x(\text{Even}(x) \wedge \text{Odd}(x^2))$

- (c) Negate the predicate logic translation. Then use a chain of logical equivalences to show that your negation is equivalent to $\forall x(\text{Even}(x) \rightarrow \text{Even}(x^2))$.

Hint: You may use the fact that $\neg\text{Odd}(a) \equiv \text{Even}(a)$.

Solution:

$\neg\exists x(\text{Even}(x) \wedge \text{Odd}(x^2))$	$\equiv \forall x\neg(\text{Even}(x) \wedge \text{Odd}(x^2))$	DeMorgan's Law for Quantifiers
	$\equiv \forall x(\neg\text{Even}(x) \vee \neg\text{Odd}(x^2))$	DeMorgan's Law
	$\equiv \forall x(\neg\text{Even}(x) \vee \text{Even}(x^2))$	Definition of Odd and Even
	$\equiv \forall x(\text{Even}(x) \rightarrow \text{Even}(x^2))$	Law of Implication

- (d) Recall that to disprove a claim, we must prove its negation. Part (c) shows us that to disprove the above claim, we should prove that if an integer x is even, then x^2 is also even. Does this match your intuition?

Solution:

Yes!

- (e) Write a proof of the fact that if an integer x is even, then x^2 is also even.

Solution:

Let x be an arbitrary integer. Suppose that x is even. Then by definition of even, there exists some integer k such that $x = 2k$.

Squaring both sides, we see that:

$$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$$

Because k is an integer, then $2k^2$ is also an integer. So by definition of even, x^2 is even.

Since x was an arbitrary integer, we can conclude that for all integers x , if x is even then x^2 is even.

- (f) Congrats, you have successfully disproved the claim!