CSE 311: Foundations of Computing I

Induction Practice Day Solutions

1. An Odd Sum

Difficulty Level: Easier

Prove by induction that the sum of the first n odd positive integers is n^2.
Hint: You may find it helpful to write the sum of the first n odd positive integers as $1 + 3 + 5 + ... + (2n - 1)$.

Solution:

Let P(n) be "$1 + 3 + 5 + ... + (2n - 1) = n^2$". We will prove P$(n)$ for all integers $n \geq 1$ by induction.

Base Case. The LHS is equal to $2(1) - 1 = 1$. The RHS is equal to $1^2 = 1$. Since $1 = 1$, the base case holds.

Inductive Hypothesis. Assume that P(k) holds true for some arbitrary integer $k \geq 1$. Then $1 + 3 + 5 + ... + (2k - 1) = k^2$

Inductive Step Goal: Show $1 + 3 + 5 + ... + (2k + 1) = (k + 1)^2$

$$\begin{aligned}
1 + 3 + 5 + ... + (2k + 1) &= 1 + 3 + 5 + ... + (2k - 1) + (2k + 1) && \text{[Show another term inside "..."]} \\
&= (k^2) + (2k + 1) && \text{[Inductive Hypothesis]} \\
&= k^2 + 2k + 1 \\
&= (k + 1)^2 && \text{[Factoring]}
\end{aligned}$$

Therefore P$(k + 1)$ holds.

Conclusion. P(n) holds for all integers $n \geq 1$ by induction.

2. Darn Divisibility

Difficulty Level: Medium

Prove by induction that for all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.

Solution:

Let $\mathsf{P}(n)$ be "$3 \mid (n^3 - n)$". We will prove $\mathsf{P}(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case. Observe that $0^3 - 0 = 0$. Since $3 \mid 0$, the base case holds.

Inductive Hypothesis. Assume that $\mathsf{P}(k)$ holds true for some arbitrary $k \in \mathbb{N}$. Then $3 \mid (k^3 - k)$. By definition of divides, there exists some integer j such that $k^3 - k = 3j$.

Inductive Step Goal: Show $3 \mid ((k+1)^3 - (k+1))$

$$\begin{aligned}
(k+1)^3 - (k+1) &= (k+1)(k^2 + 2k + 1) - k - 1 && \text{[Algebra]} \\
&= k^3 + 2k^2 + k + k^2 + 2k + 1 - k - 1 && \text{[Algebra]} \\
&= (k^3 - k) + 3k^2 + 3k && \text{[Algebra]} \\
&= 3j + 3k^2 + 3k && \text{[Inductive Hypothesis]} \\
&= 3(j + k^2 + k) && \text{[Algebra]}
\end{aligned}$$

Since j, k are integers, $j + k^2 + k$ is an integer. So by definition of divides, $3 \mid ((k+1)^3 - (k+1))$. Therefore $\mathsf{P}(k+1)$ holds.

Conclusion. $\mathsf{P}(n)$ holds for all $n \in \mathbb{N}$ by induction.

3. Geometric Sum

Difficulty Level: Medium

Suppose that a and r are real numbers with $r \neq 1$. Prove by induction that for all $n \in \mathbb{N}$:

$$a + ar + ar^2 + ... + ar^n = \frac{a \cdot r^{n+1} - a}{r - 1}$$

Solution:

Let P(n) be "$a + ar + ar^2 + ... + ar^n = \frac{a \cdot r^{n+1} - a}{r-1}$". We will prove P$(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case. The LHS simplifies to $a \cdot r^0 = a$. The RHS simplifies to $\frac{a \cdot r^{0+1} - a}{r-1} = \frac{ar-a}{r-1} = \frac{a(r-1)}{r-1} = a$. Since $a = a$, the base case holds.

Inductive Hypothesis. Assume that P(k) holds true for some arbitrary $k \in \mathbb{N}$. Then $a + ar + ar^2 + ... + ar^k = \frac{a \cdot r^{k+1} - a}{r-1}$.

Inductive Step Goal: Show $a + ar + ar^2 + ... + ar^{k+1} = \dfrac{a \cdot r^{k+2} - a}{r - 1}$

$$\begin{aligned}
a + ar + ar^2 + ... + ar^{k+1} &= a + ar + ar^2 + ... + ar^k + ar^{k+1} && \text{[Show another term inside "..."]} \\
&= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1} && \text{[Inductive Hypothesis]} \\
&= \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+1}(r-1)}{r - 1} && \text{[Finding Common Denominator]} \\
&= \frac{ar^{k+1} - a + ar^{k+1}r - ar^{k+1}}{r - 1} && \text{[Algebra]} \\
&= \frac{ar^{k+2} - a}{r - 1} && \text{[Algebra]}
\end{aligned}$$

Therefore P$(k+1)$ holds.

Conclusion. P(n) holds for all $n \in \mathbb{N}$ by induction.

4. Inequalities

Difficulty Level: Medium

Prove by induction that for all integers $n \geq 1$, we have:

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

Solution:

Let P(n) be "$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$". We will prove P($n$) for all integers $n \geq 1$ by induction.

Base Case. ($n = 1$) The LHS and RHS are both $\frac{1}{2}$. Since $\frac{1}{2} \leq \frac{1}{2}$, the base case holds.

Inductive Hypothesis. Assume that P(k) holds true for some arbitrary integer $k \geq 1$. Then $\frac{1}{2k} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)}$.

Inductive Step Goal: Show $\frac{1}{2(k+1)} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2(k+1)-1)}{2 \cdot 4 \cdot 6 \cdots (2(k+1))}$

$$\begin{aligned}
\frac{1 \cdot 3 \cdot 5 \cdots (2(k+1)-1)}{2 \cdot 4 \cdot 6 \cdots (2(k+1))} &= \frac{1 \cdot 3 \cdot 5 \cdots (2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k+2)} && \text{[Algebra]} \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)}{2 \cdot 4 \cdot 6 \cdots 2k \cdot (2k+2)} && \text{[Showing another term inside the ...]} \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k} \cdot \frac{2k+1}{2k+2} && \text{[Algebra]} \\
&\geq \frac{1}{2k} \cdot \frac{2k+1}{2k+2} && \text{[Inductive Hypothesis]} \\
&= \frac{2k+1}{2k} \cdot \frac{1}{2k+2} && \text{[Algebra]} \\
&= (1 + \frac{1}{2k}) \cdot \frac{1}{2k+2} && \text{[Algebra]} \\
&\geq \frac{1}{2k+2} && \text{[Since } \frac{1}{2k} \geq 0 \text{ so } 1 + \frac{1}{2k} \geq 1] \\
&= \frac{1}{2(k+1)} && \text{[Algebra]}
\end{aligned}$$

Therefore $\frac{1}{2(k+1)} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2(k+1)-1)}{2 \cdot 4 \cdot 6 \cdots (2(k+1))}$ so P($k+1$) holds.

Conclusion. P(n) holds for all integers $n \geq 1$ by induction.

5. Sets of Integers

Difficulty Level: Hard

Prove by induction that for all integers $n \geq 1$, given a set of $n+1$ distinct positive integers, none exceeding $2n$, there is at least one integer in the set that divides another integer in the set.

Solution:

Let $\mathsf{P}(n)$ be "Given a set of $n+1$ distinct integers between the values of 1 and $2n$ inclusive, there is at least one integer in the set that divides another integer in the set". We will prove $\mathsf{P}(n)$ for all integers $n \geq 1$ by induction.

Base Case. ($n = 1$) Consider the case of 2 integers, call them x_1 and x_2, between the values of 1 and 2 inclusive. Because the integers must be distinct, there are only two cases:
$x_1 = 1, x_2 = 2$. Then $x_1 \mid x_2$.
$x_1 = 2, x_2 = 1$. Then $x_2 \mid x_1$.

So in all cases, $\mathsf{P}(1)$ holds.

Inductive Hypothesis. Assume that $\mathsf{P}(k)$ holds true for some arbitrary integer k. Then for any set of $k+1$ distinct integers between the values of 1 and $2k$ inclusive, there is at least one integer in the set that divides another integer in the set.

Inductive Step. Consider a set of $k+2$ distinct integers between the values of 1 and $2(k+1) = 2k+2$ inclusive.

We break the proof into cases, where we consider the number of elements between 1 and $2k$ inclusive.
Case 1: $k+1$ or more of the elements lie between 1 and $2k$ inclusive. Then we can directly apply the IH to conclude that there is at least one integer in the set that divides another integer in the set.

Case 2: k or fewer of the elements lie between 1 and $2k$ inclusive. That means **at least** 2 of the elements lie between $2k+1$ and $2k+2$ inclusive. However, we are dealing with a set of numbers, which means we cannot have repeats. So there can only be 2 elements between $2k+1$ and $2k+2$; namely $2k+1$ and $2k+2$ themselves.

Case 2a: $k+1$ is in the set of numbers. Then because $k+1$ divides $2k+2$, we are done.
Case 2b: $k+1$ is **not** in the set of numbers. Denote the set of integers that we are given that fall between 1 and $2k$ as the set S. Observe that $|S| = k$. Then consider the set $S' = S \cup \{k+1\}$. Observe that $|S'| = k+1$, and S' only contains integers from 1 to $2k$ inclusive. Then we can apply the IH to this set S' to see that there are 2 integers in S', call them a and b, such that $a \mid b$.

Case 2bi: $a \neq k+1$ and $b \neq k+1$. Then both $a, b \in S$. Then we have found two elements of the set such that one divides the other, and we are done.
Case 2bii: $b = k+1$, $a \neq k+1$. Then $a \mid (k+1)$. But then $a \mid (2k+2)$ too. So there are two integers, a and $2k+2$, in our set such that a divides $2k+2$.
Case 2biii: $a = k+1$, $b \neq k+1$. Observe that $a \mid b$ where b is an integer between 1 and $2k$. But this is not possible, as the smallest multiple of $k+1$ is $2k+2$, which is out of range. So this case will not occur.

Thus in all cases, we have found one element of the set that divides another element of the set, as desired. So $\mathsf{P}(k+1)$ holds.

Conclusion. $\mathsf{P}(n)$ holds for all integers $n \geq 1$ by induction.

6. Chessboard

Difficulty Level: Hard

A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction), OR two spaces horizontally (in either direction) and one space vertically (in either direction). Suppose we have an infinite chessboard that starts at the bottom left corner $(0, 0)$, and extends upwards and to the right infinitely. The board is made up of all squares (m, n) where m, n are natural numbers that denote the column number and row number of the square, respectively. Use mathematical induction to show that a knight starting at $(0, 0)$ can visit every square using a finite sequence of moves.

Hint: Use induction on the variable $s = m + n$.

Solution:

Let P(s) be "The knight can visit any square (m, n) that is a distance of s from $(0, 0)$, where the distance s is defined as $m + n$". We will prove P(s) for all $s \in \mathbb{N}$ by induction.

Base Case. $(s = 0)$ The only square that is of distance 0 from $(0, 0)$ is the square $(0, 0)$ itself. Since the knight starts at $(0, 0)$, certainly it can reach this square.

Inductive Hypothesis. Assume that P(k) holds true for some arbitrary $k \in \mathbb{N}$. Then the knight can reach any square that is k distance from $(0, 0)$.

Inductive Step Goal: Show that the knight can reach any square $k + 1$ distance from $(0, 0)$

Consider an arbitrary square (a, b), which is $k + 1$ distance from $(0, 0)$. I.e. $a + b = k + 1$. We aim to show that the knight can reach this square.

Observe that depending on where the knight is on the chessboard, its motion may be restricted. In particular, it is possible that the knight is at the left wall of the board and cannot move further left. It is also possible that the knight is at the bottom wall of the board and cannot move further down. Note that the only location at which **both** restrictions would be present is $(0, 0)$. Since the knight is at a distance of $k + 1 \geq 1$ from $(0, 0)$, the knight is not at $(0, 0)$, so we don't have to worry about this case.

Also note that the knight may always travel to the right or up without concern of falling off the board.

Case 1: The knight is at some point (a, b) that is not at the bottom edge of the board. Then there is a square of the board directly 1 unit below the knight's current position, i.e. at $(a, b - 1)$. That square would be of distance k from $(0, 0)$, since $a + b - 1 = k + 1 - 1 = k$. So by the IH, the knight can reach the square $(a, b - 1)$. Now consider the following moves:

- The knight moves right two, up one from $(a, b - 1)$ to $(a + 2, b)$
- The knight moves left one, up two $(a + 2, b)$ to $(a + 1, b + 2)$
- The knight moves left one, down two $(a + 1, b + 2)$ to (a, b)

Thus we have shown a series of moves after which the knight reaches (a, b).

Case 2: The knight is at some point (a, b) that is not at the left edge of the board. Then there is a square of the board directly 1 to the left of the knight's current position, i.e. at $(a - 1, b)$. That square would be of distance k from $(0, 0)$, since $a - 1 + b = k + 1 - 1 = k$. So by the IH, the knight can reach the square $(a - 1, b)$. Now consider the following moves:

- The knight moves right one, up two from $(a - 1, b)$ to $(a, b + 2)$
- The knight moves right two, down one from $(a, b + 2)$ to $(a + 2, b + 1)$
- The knight moves left two, down one from $(a + 2, b + 1)$ to (a, b)

Thus we have shown a series of moves after which the knight reaches (a, b).
Thus in all cases, the knight can reach (a, b). Therefore P$(k + 1)$ holds.

Conclusion. P(s) holds for all $s \in \mathbb{N}$ by induction.