"Proof by contradiction is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game" - G. H. Hardy

Proof by Contradiction

CSE 311: Foundations of Computing I
Lecture 15

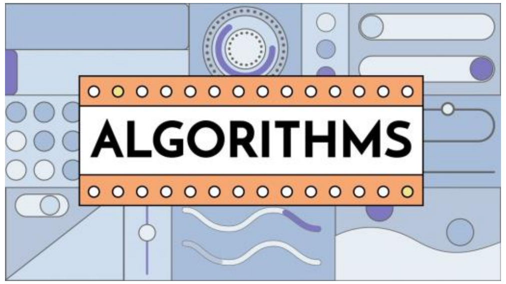
Announcements

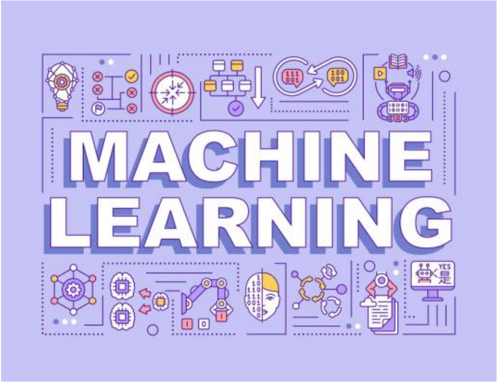
- HW5 due Wednesday at 11:59 pm.
  - Feedback before the midterm is only guaranteed if you don't use late days.
- Midterm is on Friday in class.
  - Optional review session tomorrow (Tuesday) from 3:00 – 4:20 in DEM 104. Will be recorded on Panopto.
  - Info about the exam has been published on the course website under the Exams tab. Includes problem categories and a practice exam.
- No new HW released this week.

Proof by Contradiction

Proof Strategies

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Existence Proof
- Induction
  - Weak Induction
  - Strong Induction
  - Structural Induction (coming soon!)
- Proof by Contradiction

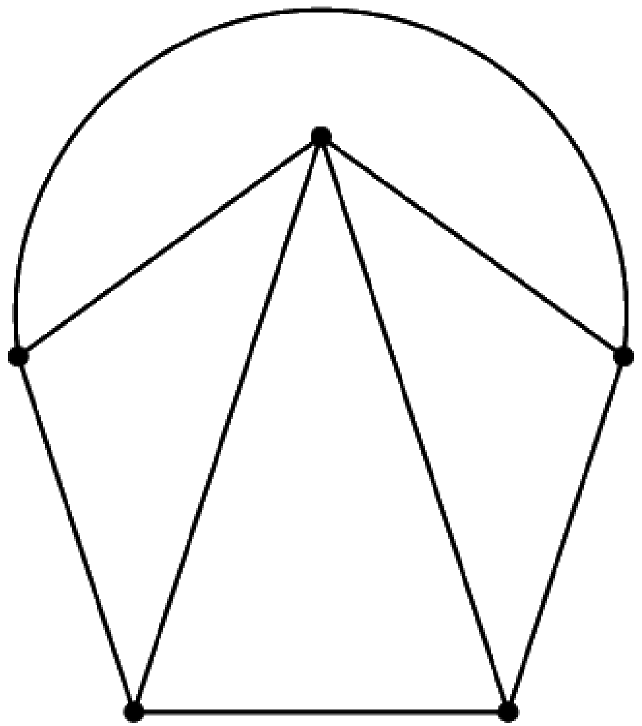




Warm-Up

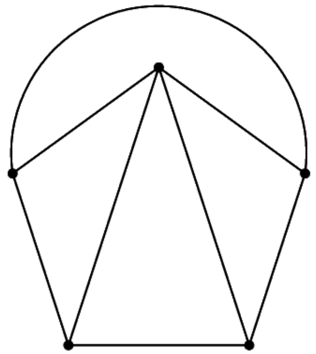
Traverse the following graph by traveling along each edge exactly once.

Any of the vertices may be selected as the starting or ending vertex.



There is no path!
Can we prove it?

Warm-Up



Claim: It is impossible to traverse this graph by traveling along each edge exactly once.

Proof:

What did we just do?

To prove that the claim p holds:

1. ____

2. ____

Why does that work?

Observe from our logical equivalences:

Proof by Contradiction

Proof by contradiction is a strategy for proving ______________________________.

- The strategy to prove p is to ____.
- E.g. the strategy to prove $p \rightarrow q$ is to ______________________________________.
- E.g. the strategy to prove $p \vee q$ is to ____.

Proof by Contradiction Skeleton

Suppose for the sake of contradiction $\neg p$.

…

Then some statement s must hold.

…

And some statement $\neg s$ must hold.

But s and $\neg s$ is a contradiction. So p must be true.

Proof by Contradiction: Remarks

- Unlike other proof techniques, we don't know *where* we're going. We're trying to find **any** contradiction. That can make it harder.
- Contradiction is a sledge-hammer.
  It can be used to prove many things. But it makes a mess.
- Use contradiction as a last-resort.

Contradiction is a sledge-hammer

Proof by Contradiction Examples

Claim 1: No integer is even and odd.

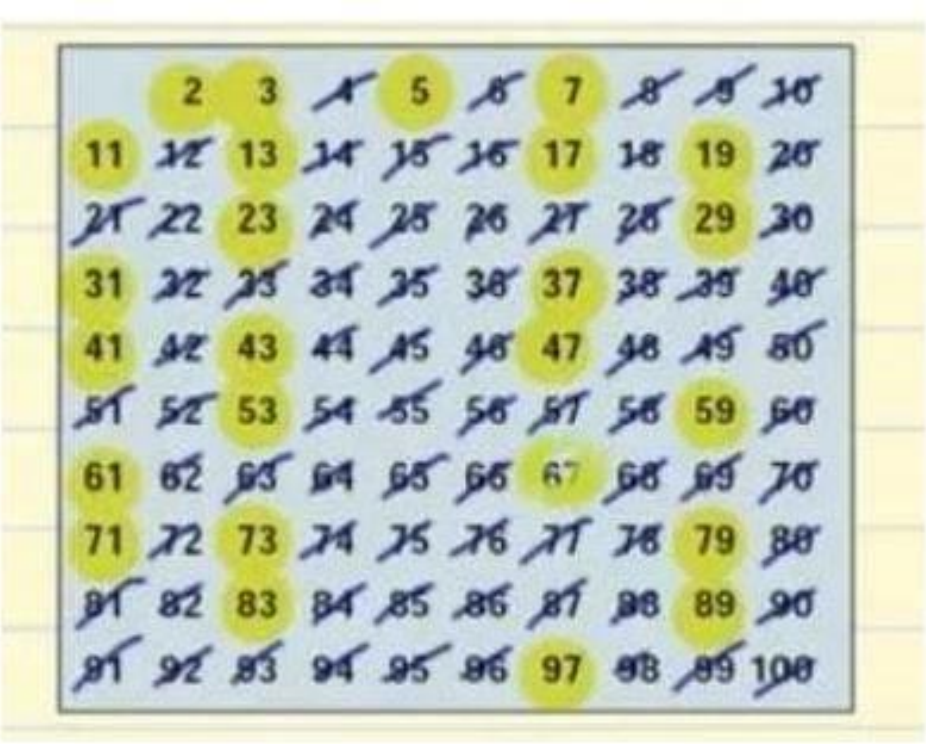
Claim 2: For all sets A, B, we have $A \cap (B \backslash \mathrm{A}) = \emptyset$.

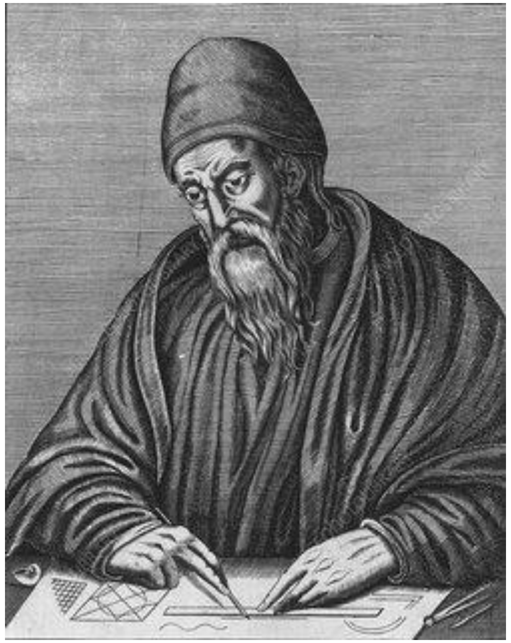
[Exercise] Claim 3: The sum of any four consecutive integers is not divisible by 4.

Euclid’s Theorem

Euclid’s Theorem: There are infinitely many prime numbers.

$2, 3, 5, 7, 11, 13, \ldots$





Euclid ~300 BC

Euclid’s Theorem: There are infinitely many prime numbers.

Discussion

Several strategies to prove $p \to q$

1) Direct Proof

2) Proof by Contrapositive

3) Proof by Contradiction

More Examples

There is no smallest positive rational number.

Recall: $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$