Strong Induction

CSE 311: Foundations of Computing I
Lecture 14

Announcements

HW5 due Wednesday at 11:59 pm

- There are 2 submission spots on Gradescope: HW5 (no late days) and HW5 (with late days)
- Feedback before the midterm is only guaranteed if you don't use late days

Midterm

- All information posted on Exams page of the course website.
- Midterm is next Friday, July 28th in class. (1 hour)
- Closed note, closed book.
- 3 reference sheets will be provided - Logical Equivalences, Number Theory, Set Theory
- One practice midterm and solutions are posted
- Optional review session this Tuesday, July 25th from 3:00 – 4:20 in DEM 104. Will be recorded on Panopto.

Midterm Topics

There will be 5 problems (with potentially multiple parts); one problem in each of these 5 categories:

- **Translation** — Translating between English & Predicate logic
- **Logic** — E.g. Equivalence Proofs, Truth Tables, CNF/DNF
- **Number Theory** — E.g. Odd & Even Proofs, Modular Arithmetic Proofs
- **Set Theory** — E.g. Set Computation, Set Proofs
- **Induction** — Ordinary Induction only, not Strong Induction

Strong Induction

Recall: Induction

Induction relied on the fact that:

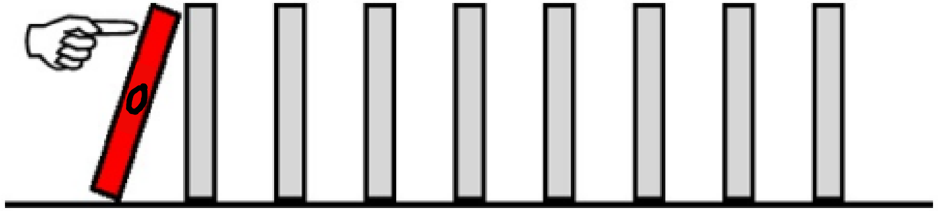
$$\forall n \, \mathrm{P}(n) \equiv \mathrm{P}(0) \land \forall k \left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)$$

<u>Base Case</u>
Prove $\mathrm{P}(0)$ holds.

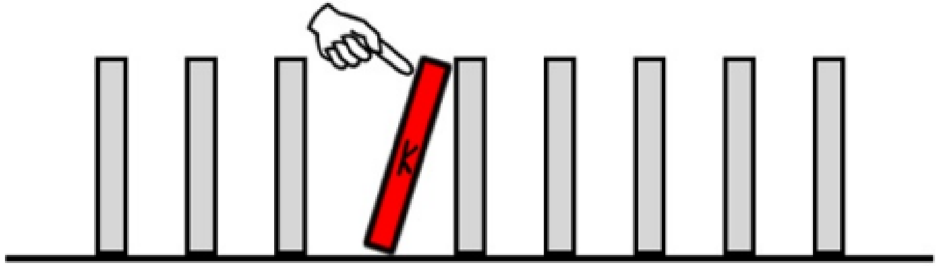
<u>Inductive Hypothesis</u>
Let $k \geq 0$ be an arbitrary integer. Suppose $\mathrm{P}(k)$ holds.

<u>Inductive Step</u>
Prove that $\mathrm{P}(k+1)$ holds (using $\mathrm{P}(k)$)

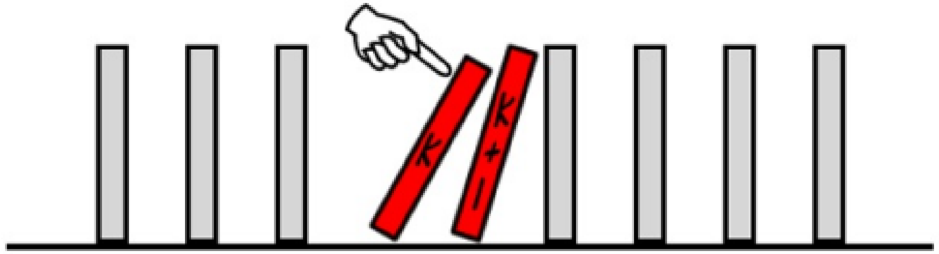
Recall: Induction



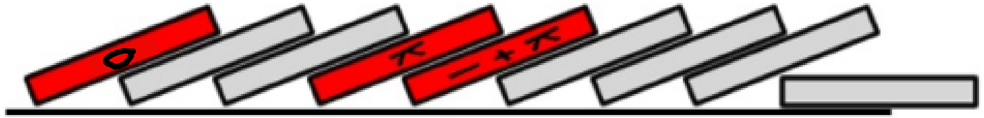
Check that the formula holds for $n = 0$



Assume the formula holds for $n = k$.



Show that the assumption *implies* that the formula holds for $n = k + 1$.



Conclude that the formula holds for all $n \in \mathbb{N}$.

Another Equivalence

There are other statements that are logically equivalent to $\forall n\ \mathrm{P}(n)$.
In particular:

$$\forall n\ \mathrm{P}(n) \equiv \mathrm{P}(0) \wedge \mathrm{P}(1) \wedge \mathrm{P}(2) \wedge \mathrm{P}(3) \ldots$$

$$\equiv \mathrm{P}(0) \wedge \big(\mathrm{P}(0) \rightarrow \mathrm{P}(1)\big) \wedge \Big(\big(\mathrm{P}(0) \wedge P(1)\big) \rightarrow \mathrm{P}(2)\Big) \wedge$$

$$\Big(\big(\mathrm{P}(0) \wedge P(1) \wedge \mathrm{P}(2)\big) \rightarrow \mathrm{P}(3)\Big) \ldots$$

$$\equiv \mathrm{P}(0) \wedge \forall k\ \Big(\big(\mathrm{P}(0) \wedge \cdots \wedge \mathrm{P}(k)\big) \rightarrow \mathrm{P}(k+1)\Big)$$

The Principle of Strong Induction

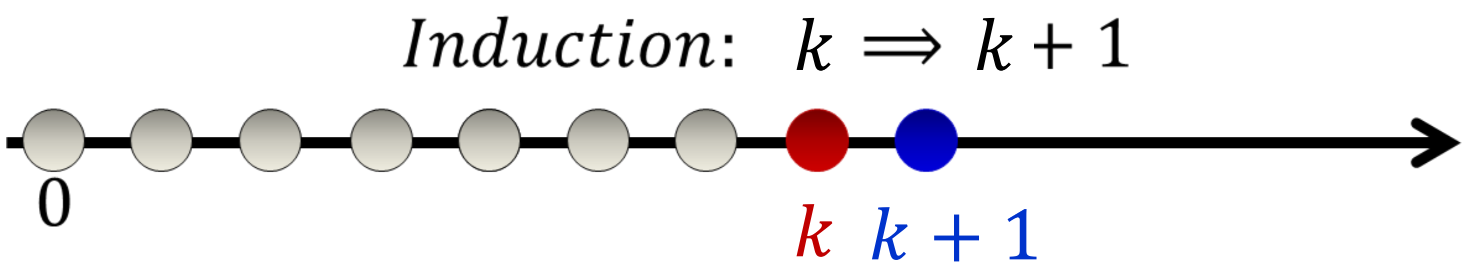
$$P(0) \land \forall k \left(\left(P(0) \land \cdots \land P(k) \right) \rightarrow P(k+1) \right)$$

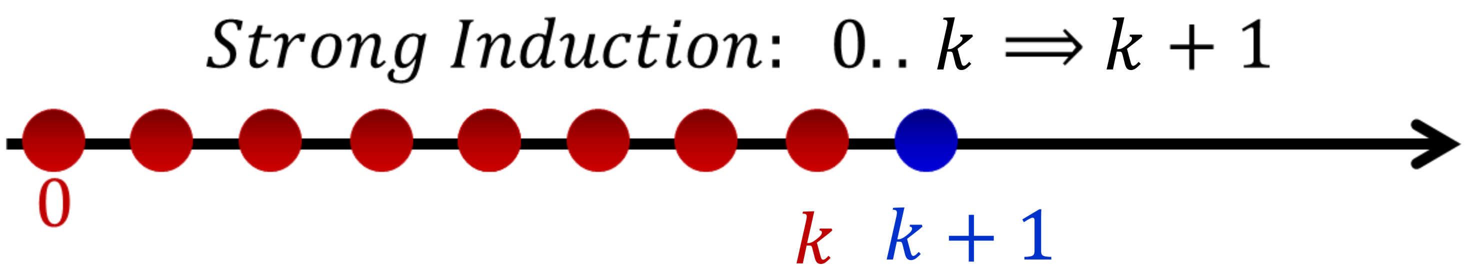
<u>Base Case</u>
Prove $P(0)$ holds.

<u>Inductive Hypothesis</u>
Let $k \geq 0$ be an arbitrary integer. Suppose $P(0) \land \cdots \land P(k)$ hold.

<u>Inductive Step</u>
Prove that $P(k+1)$ holds

Strong Induction





Fundamental Theorem of Arithmetic

Theorem: Every positive integer greater than 1 has a unique prime factorization.

$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$591 = 3 \cdot 197$$

Let's prove that a factorization into primes **exists** using induction (uniqueness is harder).

[Incorrect Proof by Induction] Prove that every positive integer greater than 1 can be written as a product of primes.

1. Let $\mathrm{P}(n)$ be "n can be written as a product of primes". We prove $\mathrm{P}(n)$ for all integers $n \geq 2$ by induction.
2. Base Case: 2 is a product of one prime (itself). Thus $\mathrm{P}(2)$ is true.
3. IH: Suppose $\mathrm{P}(k)$ hold for an arbitrary integer $k \geq 2$. I.e. k can be written as the product of primes.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that $k+1$ can be written as a product of primes.
   Case 1: $k+1$ is prime. Then $k+1$ is automatically the product of primes.

   Case 2: $k+1$ is composite. How can we use $\mathrm{P}(k)$ here?? We're stuck.

[Proof by Strong Induction] Prove that every positive integer greater than 1 can be written as a product of primes.

1. Let $\mathrm{P}(n)$ be "n can be written as a product of primes". We prove $\mathrm{P}(n)$ for all integers $n \geq 2$ by strong induction.
2. Base Case: 2 is a product of one prime (itself). Thus $\mathrm{P}(2)$ is true.
3. IH: Suppose $\mathrm{P}(2) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $k \geq 2$. That is, $2, 3, 4, \ldots, k$ can all be written as the product of primes.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that $k+1$ can be written as a product of primes. Case 1: $k+1$ is prime. Then $k+1$ is automatically the product of primes.

   Case 2: $k+1$ is composite. Then we can write $k+1 = a \cdot b$ for some divisors a, b such that $2 \leq a < k+1$ and $2 \leq b < k+1$. By the IH, a and b can be written as the product of primes. So $a = p_1 p_2 \cdots p_s$ and $b = q_1 q_2 \cdots q_t$ for some primes $p_1, \ldots, p_s, q_1, \ldots q_t$. Thus $k+1 = ab = p_1 \cdots p_s \cdot q_1 \cdots q_t$ which is a product of primes.

   So in any case, $\mathrm{P}(k+1)$ holds.
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 2$ by strong induction.

Strong Induction vs. Weak Induction

- “Normal” Induction is otherwise known as Weak Induction
- All induction proofs could be written by Strong Induction instead. It’s a *stronger* hypothesis to use. There is more to work with.
- However, there’s often the philosophy to only use a stronger hypothesis when needed.

Strong Induction Example

Stamp Collection

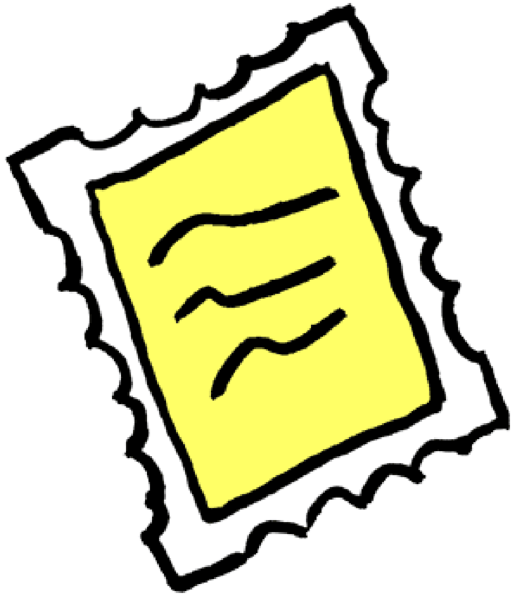
Stamp Collection

I have a collection of $4¢$ and $5¢$ stamps. Prove that for all $n \geq 12$, I can make $n¢$ worth of stamps.

Examples:

$13¢ = 5¢ + 4¢ + 4¢$

$22¢ = 5¢ + 5¢ + 4¢ + 4¢ + 4¢$



[Attempted Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make n ¢ worth of stamps.

1. Let $\mathrm{P}(n)$ be "I can make n ¢ worth of stamps with just 4¢ and 5¢ stamps." We prove $\mathrm{P}(n)$ for all integers $n \geq 12$ by strong induction.
2. Base Case: 12¢ can be made with three 4¢ stamps. Thus $\mathrm{P}(12)$ is true.
3. IH: Suppose $\mathrm{P}(12) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $k \geq 12$. I.e. we can make 12¢, 13¢, …, k¢ worth of stamps with just 4¢ and 5¢ stamps.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that we can make $k+1$ cents in stamps. By the IH, we can make $k-3$ cents in stamps. Adding another 4¢ stamp gives exactly $k+1$ cents.
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 12$ by strong induction.

WRONG

What was the problem?

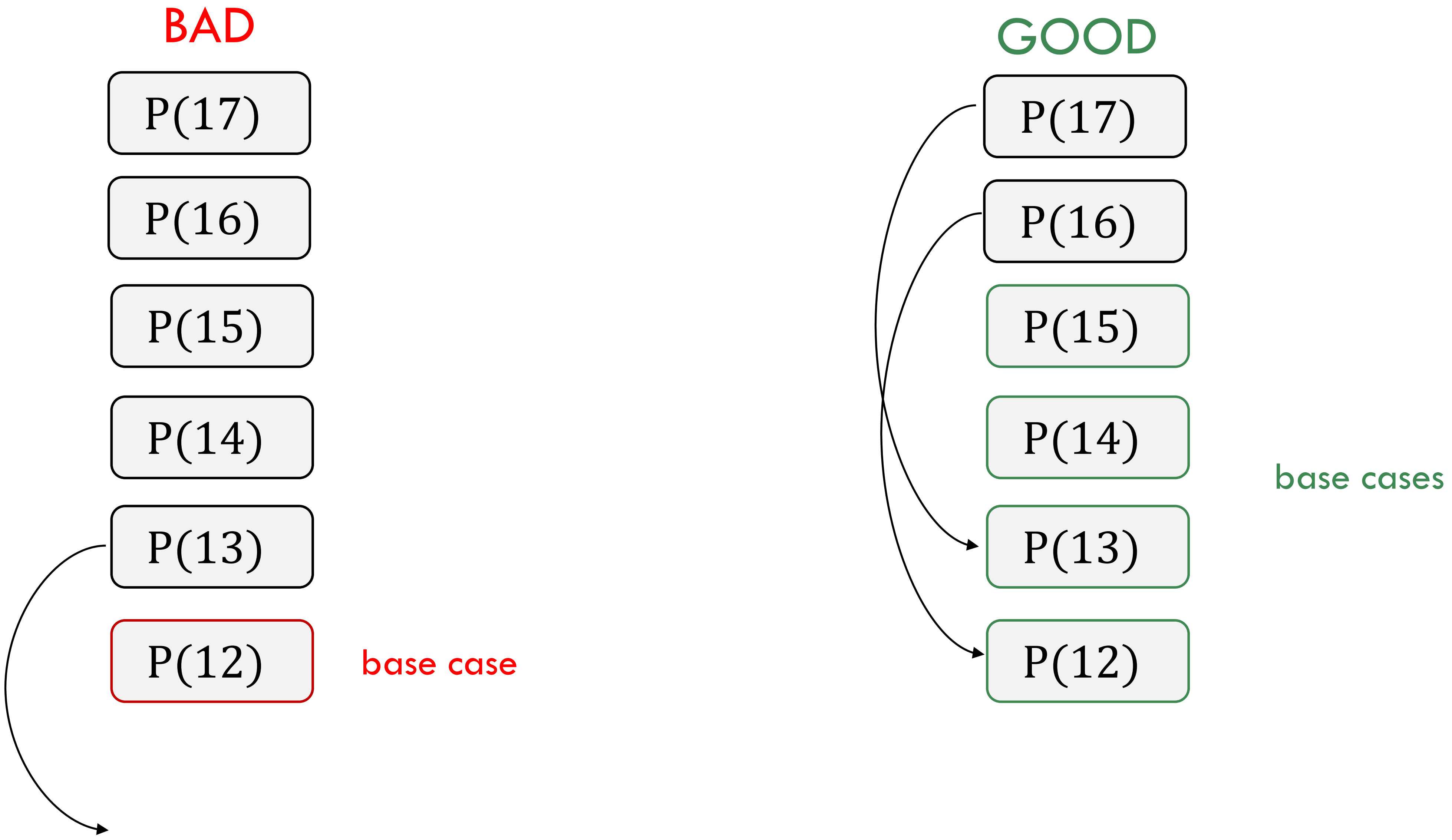
We don't know $\mathrm{P}(13)$ holds.

When $k = 12$, and $k + 1 = 13$:

- Our IH assumes just $\mathrm{P}(12)$
- In the IS, we say since $\mathrm{P}(9)$ holds (going back to $k - 3$), then $\mathrm{P}(13)$ holds.
- But we don't know anything about $\mathrm{P}(9)$! It might not even be true!

Lesson: If we go back s steps in the IS, we need s base cases.

Tower Visualization



[Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make n ¢ worth of stamps.

1. Let $\mathrm{P}(n)$ be "I can make n ¢ worth of stamps with just 4¢ and 5¢ stamps." We prove $\mathrm{P}(n)$ for all integers $n \geq 12$ by strong induction.
2. Base Cases:
   12¢ can be made with three 4¢ stamps. Thus $\mathrm{P}(12)$ is true.
   13¢ can be made with two 4¢ stamps and one 5¢ stamps. Thus $\mathrm{P}(13)$ is true.
   14¢ can be made with one 4¢ stamp and two 5¢ stamps. Thus $\mathrm{P}(14)$ is true.
   15¢ can be made with three 5¢ stamps. Thus $\mathrm{P}(15)$ is true.
3. IH: Suppose $\mathrm{P}(12) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $\boldsymbol{k \geq 15}$. I.e. we can make 12¢, 13¢, …, k¢ worth of stamps with just 4¢ and 5¢ stamps.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that we can make $k+1$ cents in stamps. By the IH, we can make $k-3$ cents in stamps. Adding another 4¢ stamp gives exactly $k+1$ cents.
   [Note: Now $k+1 \geq 16$, so $k-3 \geq 12$. We're in the clear!]
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 12$ by strong induction.

Strong Induction Lesson

Be careful about base cases!!

Strong Induction Template

Strong Induction Template

1. Define $\mathrm{P}(n)$. State that your proof is by strong induction on n.
2. Base Case: Show your base cases $\mathrm{P}(b_{\min}), \ldots, \mathrm{P}(b_{\max})$ are true.
3. Inductive Hypothesis: Suppose $\mathrm{P}(b_{\min}) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $k \geq b_{\max}$.
4. Inductive Step: Prove $\mathrm{P}(k+1)$ using the IH.
5. Conclusion: Conclude by saying $\mathrm{P}(n)$ holds for all integers $n \geq b_{\min}$ by strong induction.

Practical Tip

- If you aren't sure how many steps you'll go back, leave space for the base cases.
- Do the IH / IS, and then fill in the base cases later.

Strong Induction Example

Fibonacci Sequence

Fibonacci Numbers

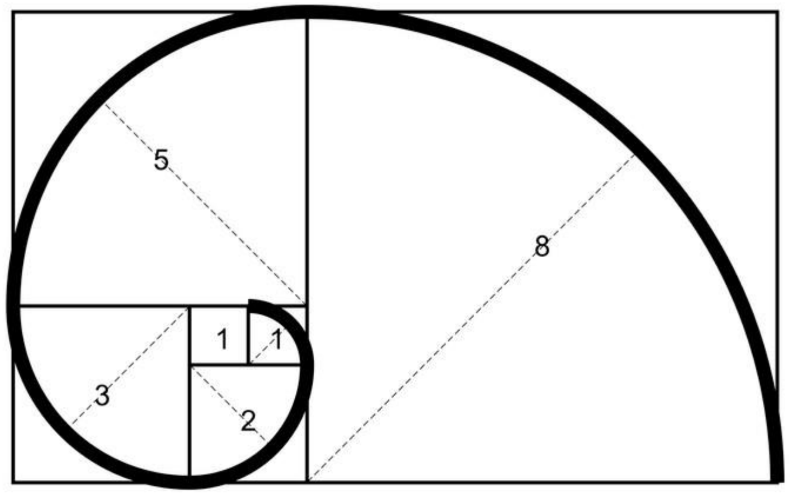
The Fibonacci Numbers are defined as follows:

$f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

I.e. $0, 1, 1, 2, 3, 5, 8, \ldots$



Fibonacci Numbers Claim

We claim that $f_n < 2^n$ for all $n \geq 0$.

$f_0 = 0$	$2^0 = 1$
$f_1 = 1$	$2^1 = 2$
$f_2 = 1$	$2^2 = 4$
$f_3 = 2$	$2^3 = 8$
$f_4 = 3$	$2^4 = 16$

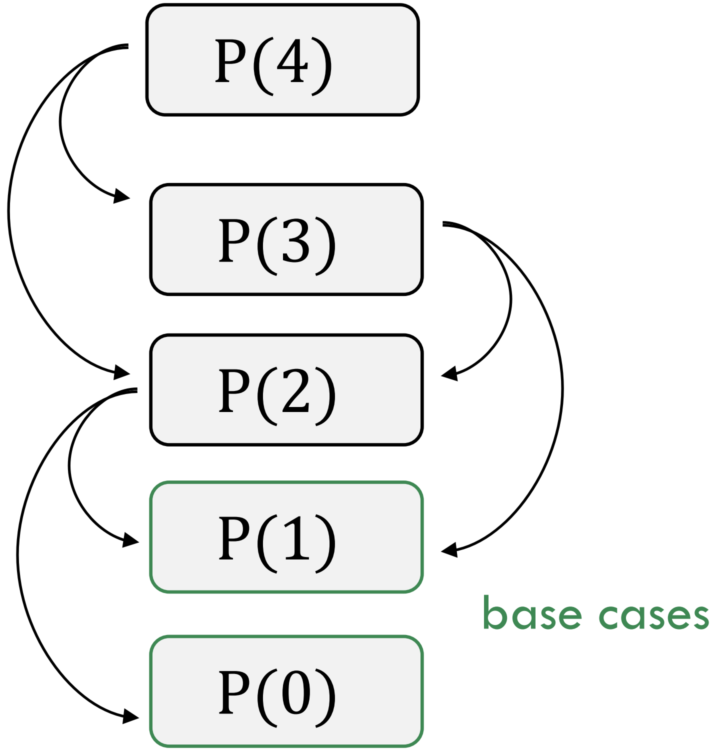
We prove by strong induction!

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

1. Let $\mathrm{P}(n)$ be "$f_n < 2^n$" We prove $\mathrm{P}(n)$ for all $n \in \mathbb{N}$ by strong induction.
2. Base Cases:
   $f_0 = 0$ and $2^0 = 1$. Since $0 < 1$, $\mathrm{P}(0)$ holds.
   $f_1 = 1$ and $2^1 = 2$. Since $1 < 2$, $\mathrm{P}(1)$ holds.
3. IH: Suppose $\mathrm{P}(0) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $\boldsymbol{k \geq 1}$.
4. IS: We aim to show $\mathrm{P}(k+1)$, i.e. that $f_{k+1} < 2^{k+1}$. Observe:
   $f_{k+1} = f_k + f_{k-1}$ Since $k + 1 \geq 2$
   $\leq 2^k + f_{k-1}$ By IH, since $\mathrm{P}(k)$ is assumed
   $\leq 2^k + 2^{k-1}$ By IH, since $\mathrm{P}(k-1)$ is assumed
   $\leq 2^k + 2^k$ Since $2^{k-1} = \frac{1}{2} \cdot 2^k \leq 2^k$
   $= 2^{k+1}$
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all $n \in \mathbb{N}$ by strong induction.

Fibonacci Tower



How many base cases?

- Always at least one base case.
- If you’re analyze a recursive function, at least one for each base case of the function.
- If you go back s steps in the proof, at least s base cases.