Strong Induction

CSE 311: Foundations of Computing I
Lecture 14

Announcements

HW5 due Wednesday at 11:59 pm

- There are 2 submission spots on Gradescope:
  HW5 (no late days) and HW5 (with late days)
- Feedback before the midterm is only guaranteed if you don't use late days

Midterm

- All information posted on Exams page of the course website.
- Midterm is next Friday, July 28th in class. (1 hour)
- Closed note, closed book.
- 3 reference sheets will be provided - Logical Equivalences, Number Theory, Set Theory
- One practice midterm and solutions are posted
- Optional review session this Tuesday, July 25th from 3:00 – 4:20 in DEM 104. Will be recorded on Panopto.

Midterm Topics

There will be 5 problems (with potentially multiple parts); one problem in each of these 5 categories:

- **Translation** — Translating between English & Predicate logic
- **Logic** — E.g. Equivalence Proofs, Truth Tables, CNF/DNF
- **Number Theory** — E.g. Odd & Even Proofs, Modular Arithmetic Proofs
- **Set Theory** — E.g. Set Computation, Set Proofs
- **Induction** — Ordinary Induction only, not Strong Induction

Strong Induction

Recall: Induction

Induction relied on the fact that:

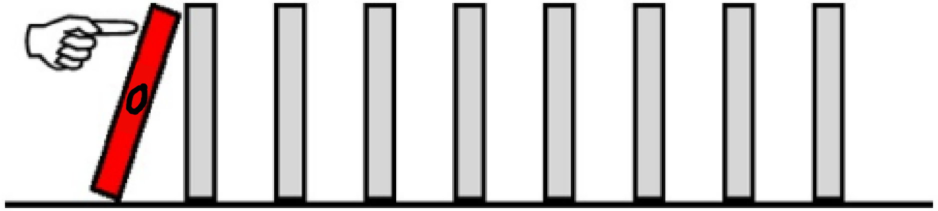
$$\forall n\, \mathrm{P}(n) \equiv \mathrm{P}(0) \wedge \forall k\, \big(\mathrm{P}(k) \to \mathrm{P}(k+1)\big)$$

<u>Base Case</u>
Prove $\mathrm{P}(0)$ holds.

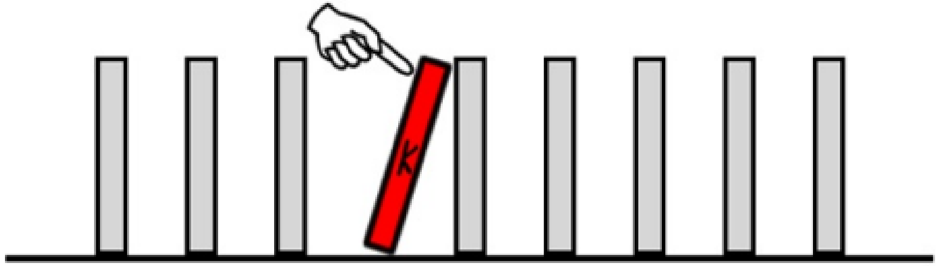
<u>Inductive Hypothesis</u>
Let $k \geq 0$ be an arbitrary integer.
Suppose $\mathrm{P}(k)$ holds.

<u>Inductive Step</u>
Prove that $\mathrm{P}(k+1)$ holds (using $\mathrm{P}(k)$)

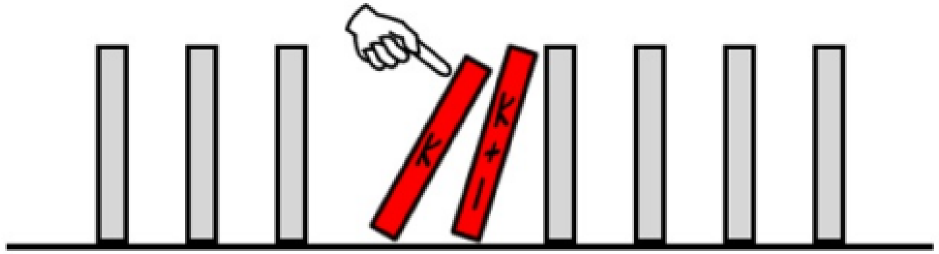
Recall: Induction



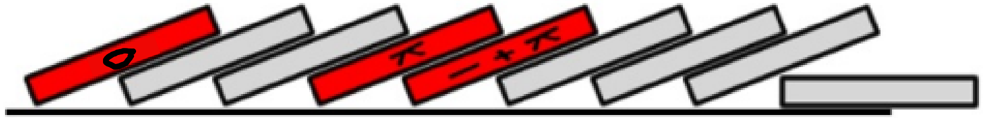
Check that the formula holds for $n = 0$



Assume the formula holds for $n = k$.



Show that the assumption *implies* that the formula holds for $n = k + 1$.



Conclude that the formula holds for all $n \in \mathbb{N}$.

Another Equivalence

There are other statements that are logically equivalent to $\forall n\ P(n)$.
In particular:

$\forall n\ P(n) \equiv$

The Principle of Strong Induction

$$P(0) \wedge \forall k \left(\left(P(0) \wedge \cdots \wedge P(k) \right) \rightarrow P(k+1) \right)$$

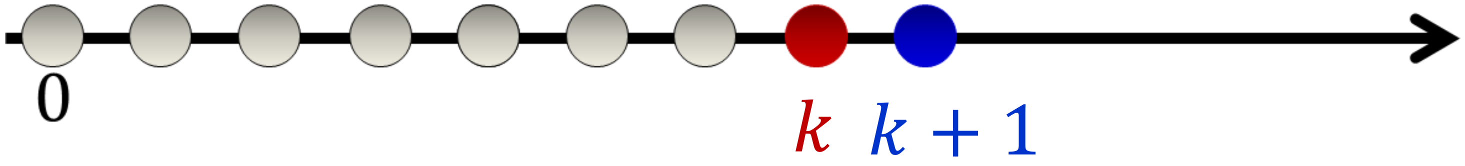
<u>Base Case</u>
Prove $P(0)$ holds.

<u>Inductive Hypothesis</u>
Let $k \geq 0$ be an arbitrary integer. Suppose $P(0) \wedge \cdots \wedge P(k)$ hold.

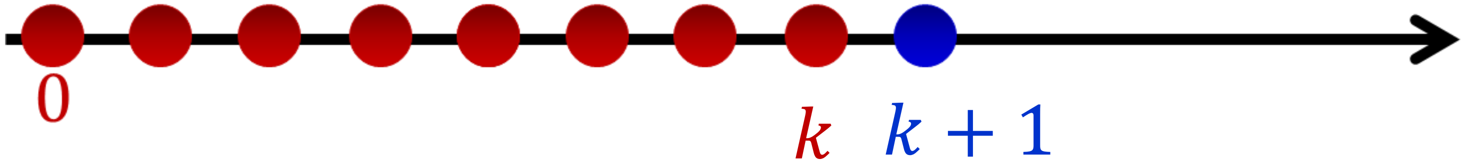
<u>Inductive Step</u>
Prove that $P(k+1)$ holds

Strong Induction

$$\textit{Induction}: \quad k \Longrightarrow k+1$$



$$\textit{Strong Induction}: \quad 0..\,k \Longrightarrow k+1$$



Fundamental Theorem of Arithmetic

Theorem: Every positive integer greater than 1 has a unique prime factorization.

$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$591 = 3 \cdot 197$$

Let's prove that a factorization into primes **exists** using induction (uniqueness is harder).

[Incorrect Proof by Induction] Prove that every positive integer greater than 1 can be written as a product of primes.

1. Let $\mathrm{P}(n)$ be

2. Base Case:

3. IH:

4. IS:

5. Conclusion:

[Proof by Strong Induction] Prove that every positive integer greater than 1 can be written as a product of primes.

1. Let $\mathrm{P}(n)$ be "n can be written as a product of primes". We prove $\mathrm{P}(n)$ for all integers $n \geq 2$ by strong induction.
2. Base Case: 2 is a product of one prime (itself). Thus $\mathrm{P}(2)$ is true.
3. IH:
4. IS:
5. Conclusion:

Strong Induction vs. Weak Induction

- “Normal” Induction is otherwise known as Weak Induction
- All induction proofs could be written by Strong Induction instead. It’s a *stronger* hypothesis to use. There is more to work with.
- However, there’s often the philosophy to only use a stronger hypothesis when needed.

Strong Induction Example

Stamp Collection

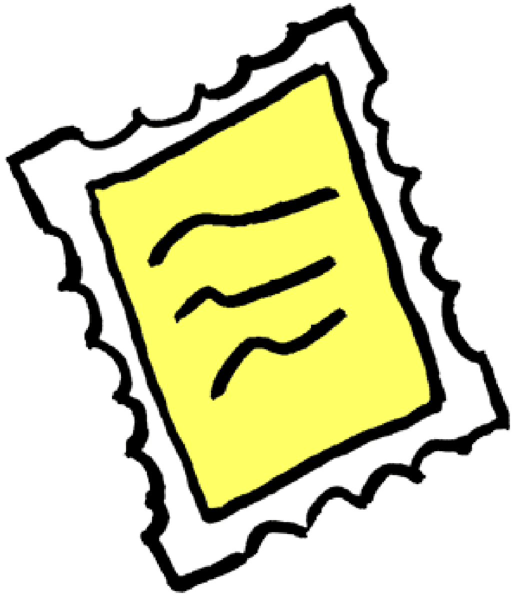
Stamp Collection

I have a collection of $4¢$ and $5¢$ stamps. Prove that for all $n \geq 12$, I can make $n¢$ worth of stamps.

Examples:

$13¢$

$22¢$



[Attempted Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make n ¢ worth of stamps.

1. Let $\mathrm{P}(n)$ be

2. Base Case:

3. IH:

4. IS:

5. Conclusion:

WRONG

What was the problem?

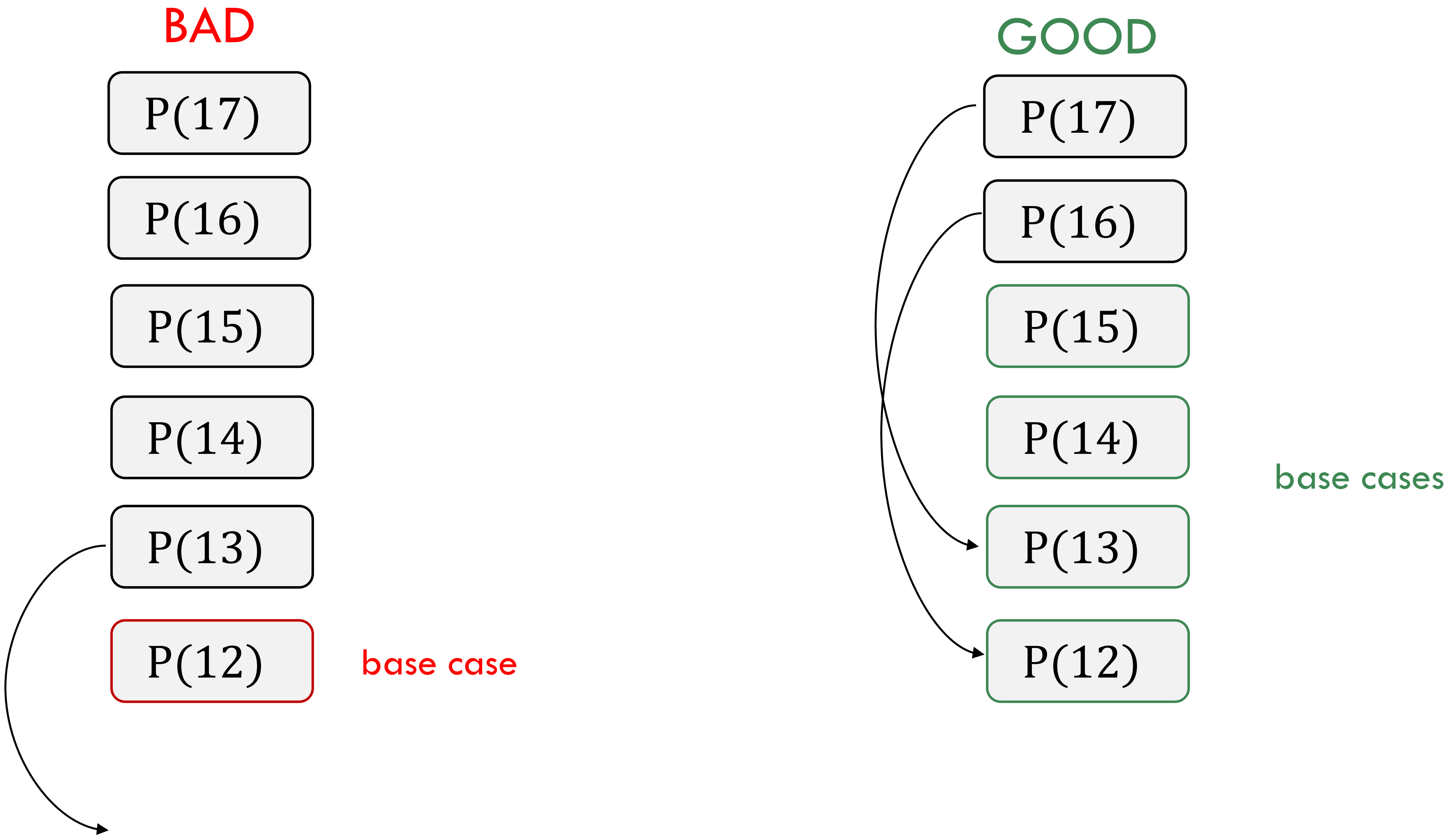
We don't know $\mathrm{P}(13)$ holds.

When $k = 12$, and $k + 1 = 13$:

- Our IH assumes just $\mathrm{P}(12)$
- In the IS, we say since $\mathrm{P}(9)$ holds (going back to $k - 3$), then $\mathrm{P}(13)$ holds.
- But we don't know anything about $\mathrm{P}(9)$! It might not even be true!

Lesson: If we go back s steps in the IS, we need s base cases.

Tower Visualization



[Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make n ¢ worth of stamps.

1. Let $\mathrm{P}(n)$ be

2. Base Cases:

3. IH:

4. IS:

5. Conclusion:

Strong Induction Lesson

Be careful about base cases!!

Strong Induction Template

Strong Induction Template

1. Define $\mathrm{P}(n)$. State that your proof is by strong induction on n.
2. Base Case: Show your base cases $\mathrm{P}(b_{\min}), \ldots, \mathrm{P}(b_{\max})$ are true.
3. Inductive Hypothesis: Suppose $\mathrm{P}(b_{\min}) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $k \geq b_{\max}$.
4. Inductive Step: Prove $\mathrm{P}(k+1)$ using the IH.
5. Conclusion: Conclude by saying $\mathrm{P}(n)$ holds for all integers $n \geq b_{\min}$ by strong induction.

Practical Tip

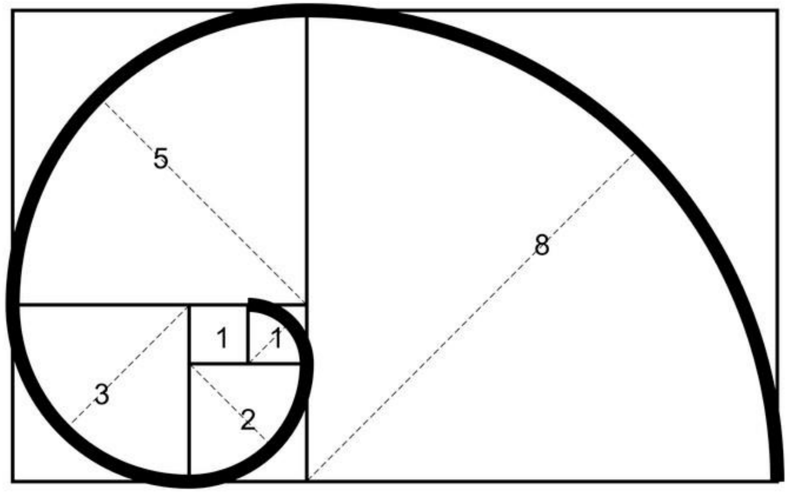
- If you aren’t sure how many steps you’ll go back, leave space for the base cases.
- Do the IH / IS, and then fill in the base cases later.

Strong Induction Example

Fibonacci Sequence

Fibonacci Numbers

The Fibonacci Numbers are defined as follows:



Fibonacci Numbers Claim

We claim that $f_n < 2^n$ for all $n \geq 0$.

$f_0 = 0$	$2^0 = 1$
$f_1 = 1$	$2^1 = 2$
$f_2 = 1$	$2^2 = 4$
$f_3 = 2$	$2^3 = 8$
$f_4 = 3$	$2^4 = 16$

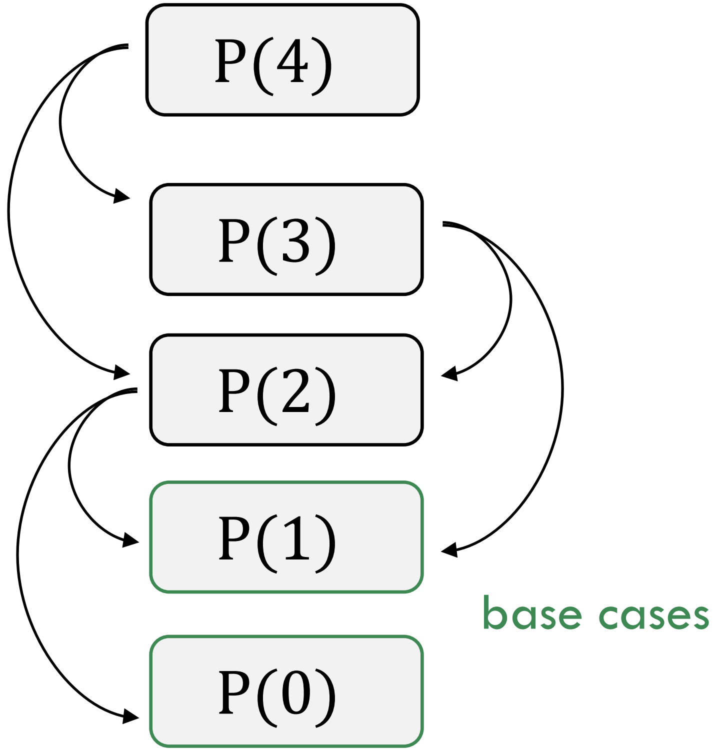
We prove by strong induction!

Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

1. Let $\mathrm{P}(n)$ be
2. Base Case(s):
3. IH:
4. IS:
5. Conclusion:

Fibonacci Tower



How many base cases?

- Always at least one base case.
- If you're analyze a recursive function, at least one for each base case of the function.
- If you go back s steps in the proof, at least s base cases.