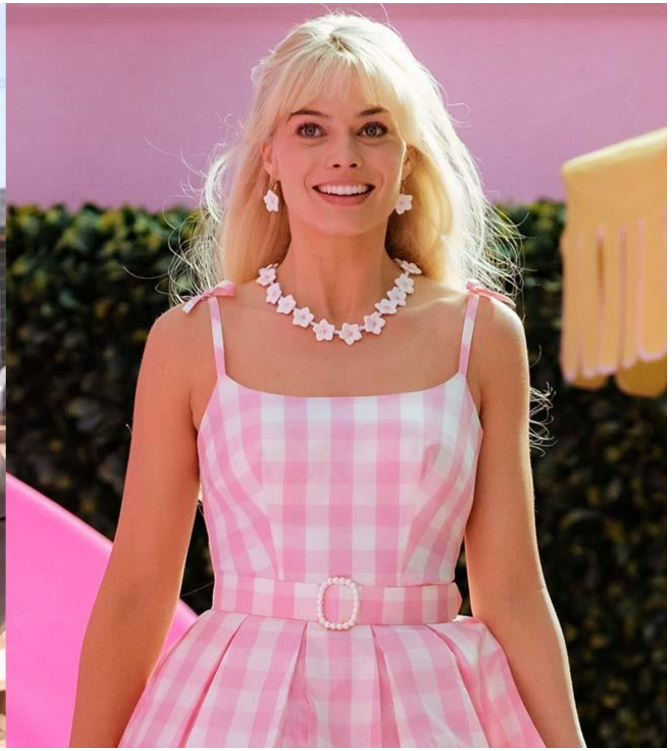
Oppenheimer or Barbie?

Poll Everywhere ☺
**pollev.com/anjalia**



# Strong Induction

CSE 311: Foundations of Computing I
Lecture 14

# Announcements

HW5 due Wednesday at 11:59 pm

- There are 2 submission spots on Gradescope:
  HW5 (no late days) and HW5 (with late days)
- Feedback before the midterm is only guaranteed if you don't use late days

# Midterm

- All information posted on Exams page of the course website.
- Midterm is next Friday, July 28th in class. (1 hour)
- Closed note, closed book.
- 3 reference sheets will be provided - Logical Equivalences, Number Theory, Set Theory
- One practice midterm and solutions are posted
- Optional review session this Tuesday, July 25th from 3:00 – 4:20 in DEM 104. Will be recorded on Panopto.

# Midterm Topics

There will be 5 problems (with potentially multiple parts); one problem in each of these 5 categories:

- **Translation** Translating between English & Predicate logic
- **Logic** E.g. Equivalence Proofs, Truth Tables, CNF/DNF
- **Number Theory** E.g. Odd & Even Proofs, Modular Arithmetic Proofs
- **Set Theory** E.g. Set Computation, Set Proofs
- **Induction** Ordinary Induction only, not Strong Induction

# Strong Induction

# Recall: Induction

Induction relied on the fact that:

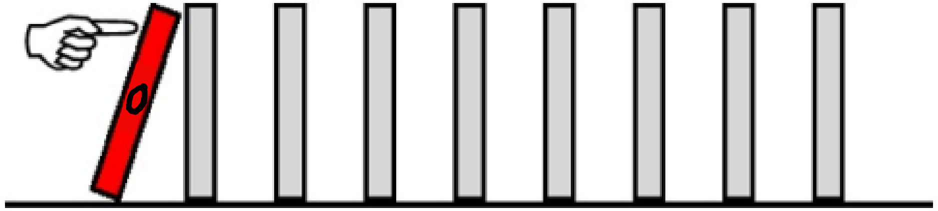
$$\forall n\, \mathrm{P}(n) \equiv \mathrm{P}(0) \wedge \forall k \left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)$$

**Base Case**
Prove $\mathrm{P}(0)$ holds.

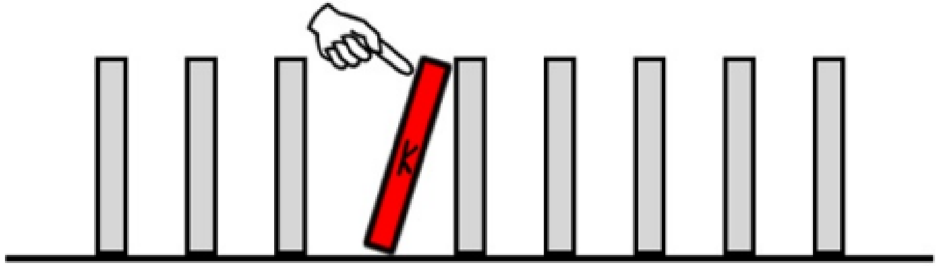
**Inductive Hypothesis**
Let $k \geq 0$ be an arbitrary integer.
Suppose $\mathrm{P}(k)$ holds.

**Inductive Step**
Prove that $\mathrm{P}(k+1)$ holds (using $\mathrm{P}(k)$)

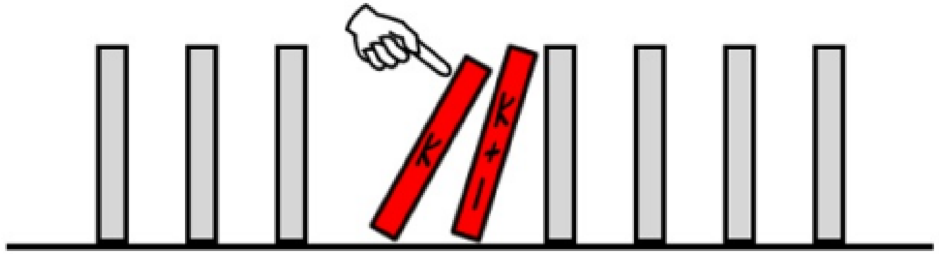
# Recall: Induction



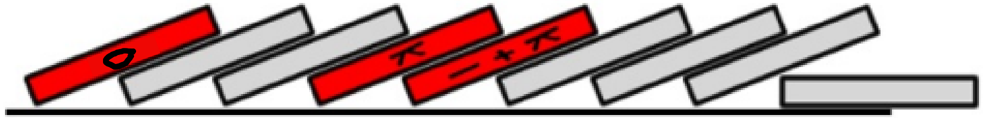

Check that the formula holds for $n = 0$




*Assume* the formula holds for $n = k$.

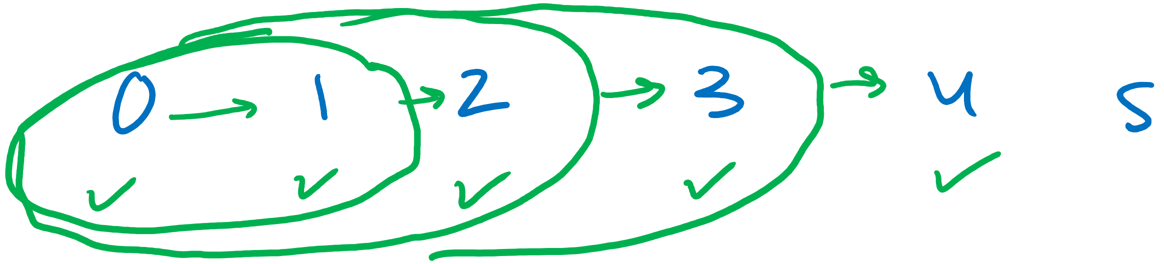



Show that the assumption *implies* that the formula holds for $n = k + 1$.




Conclude that the formula holds for all $n \in \mathbb{N}$.

# Another Equivalence




There are other statements that are logically equivalent to $\forall n\, P(n)$.
In particular:

$$\forall n\, P(n) \equiv P(0) \land P(1) \land P(2) \land P(3) \land \ldots$$

$$\equiv P(0) \land (P(0) \to P(1)) \land ((P(0) \land P(1)) \to P(2)) \land ((P(0) \land P(1) \land P(2)) \to P(3)) \land \ldots$$

$$\equiv \underline{P(0)} \land \forall k\, (\underline{(P(0) \land P(1) \land \ldots \land P(k))} \to \underline{P(k+1)})$$

# The Principle of Strong Induction

$$P(0) \wedge \forall k \left(\left(P(0) \wedge \cdots \wedge P(k)\right) \rightarrow P(k+1)\right)$$

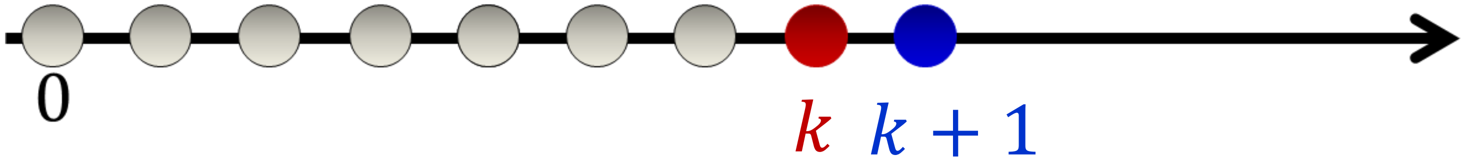
**Base Case**
Prove $P(0)$ holds.

**Inductive Hypothesis**
Let $k \geq 0$ be an arbitrary integer. Suppose $P(0) \wedge \cdots \wedge P(k)$ hold.

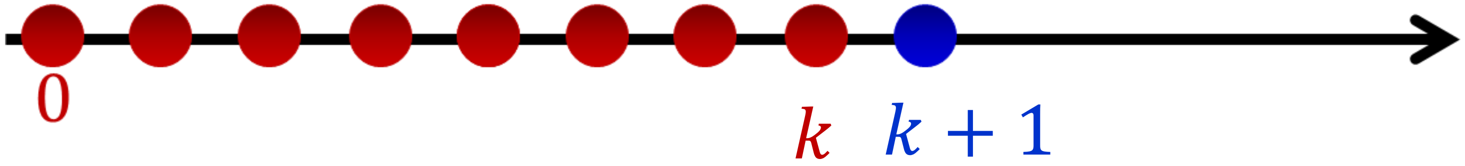
**Inductive Step**
Prove that $P(k+1)$ holds

# Strong Induction

$Induction: \; k \Longrightarrow k+1$




$Strong\ Induction: \; 0..\,k \Longrightarrow k+1$



# Fundamental Theorem of Arithmetic

Theorem: Every positive integer greater than $1$ has a unique prime factorization.

$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$591 = 3 \cdot 197$$

Let's prove that a factorization into primes **exists** using induction (uniqueness is harder).

[Incorrect Proof by Induction] Prove that every positive integer greater than 1 can be written as a product of primes.

1. Let $P(n)$ be "n can be written as a product of primes." We prove P(n) for all integers $n \geq 2$ by induction.

2. Base Case: $n = 2$. 2 is a product of one prime (itself). So P(2) holds.

3. IH: Suppose P(k) holds for an arbitrary integer $k \geq 2$. i.e. k can be written as a product of primes.

4. IS: Goal: P(k+1) i.e. k+1 can be written as product of primes.

STUCK

$k = p_1 p_2 \cdots p_t$

$k+1 = \underline{\qquad\qquad}$

$20 = 2 \cdot 2 \cdot 5$

$21 = 3 \cdot 7$

5. Conclusion:

[Proof by Strong Induction] Prove that every positive integer greater than 1 can be written as a product of primes.

1. Let $P(n)$ be "$n$ can be written as a product of primes". We prove $P(n)$ for all integers $n \geq 2$ by strong induction.
2. Base Case: 2 is a product of one prime (itself). Thus $P(2)$ is true.
3. IH: suppose $P(2) \wedge P(3) \wedge \ldots \wedge P(k)$ hold for an arbitrary integer $k \geq 2$.
4. IS: Goal $P(k+1)$

   Case 1: $k+1$ is prime. Then certainly $k+1$ can be written as a product of primes.

   Case 2: $k+1$ is composite. Then $k+1 = a \cdot b$ for $2 \leq a, b < k+1$.

   Then by IH, $a = p_1 p_2 \cdots p_t$ and $b = q_1 \cdot q_2 \cdots q_s$ for primes $p_1, \ldots, p_t$ and $q_1, \ldots, q_s$. Thus $k+1 = ab = p_1 \cdots p_t \cdot q_1 \cdots q_s$. So $k+1$ is a product of primes. Thus $P(k+1)$ holds.
5. Conclusion: Thus $P(n)$ holds for all integers $n \geq 2$ by strong induction.

# Strong Induction vs. Weak Induction

- “Normal” Induction is otherwise known as Weak Induction
- All induction proofs could be written by Strong Induction instead. It’s a *stronger* hypothesis to use. There is more to work with.
- However, there’s often the philosophy to only use a stronger hypothesis when needed.

# Strong Induction Example

Stamp Collection

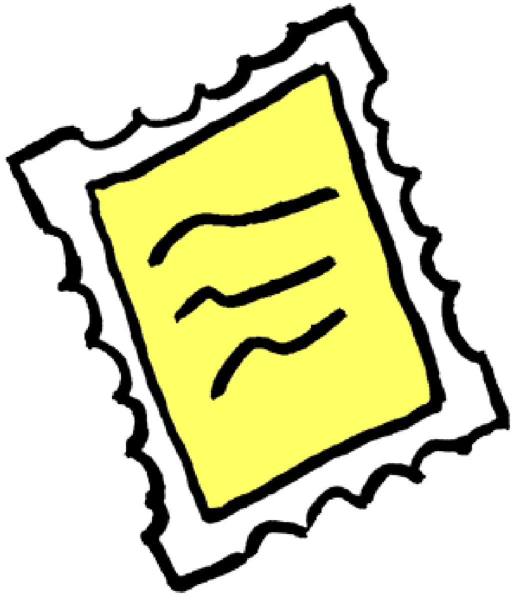
# Stamp Collection

I have a collection of $4¢$ and $5¢$ stamps. Prove that for all $n \geq 12$, I can make $n¢$ worth of stamps.

Examples:

$13¢$ 4¢, 4¢, 5¢

$22¢$ 5¢, 5¢, 4¢, 4¢, 4¢



[Attempted Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make $n$ ¢ worth of stamps.

k+1 = 13

k−3 = 9

1. Let P($n$) be "I can make n ¢ worth of stamps using 4 ¢ and 5¢ stamps." We prove P(n) for all integers n ≥ 12 by strong induction.
2. Base Case:
   12 ¢ can be made with 3 × 4 ¢. So P(12) holds. ✓
3. IH: Suppose P(12) ∧ P(13) ∧ ... ∧ P(k) for an arbitrary integer k ≥ 12. i.e. we can make 12 ¢, 13 ¢, ......, k ¢.
4. IS: Goal: P(k+1)
   By IH, we can make k−3 ¢. Add 4 ¢ stamp to make k+1 ¢. Thus P(k+1) holds.
5. Conclusion: Thus P(n) holds for all int n ≥ 12 by induction.

WRONG

# What was the problem?

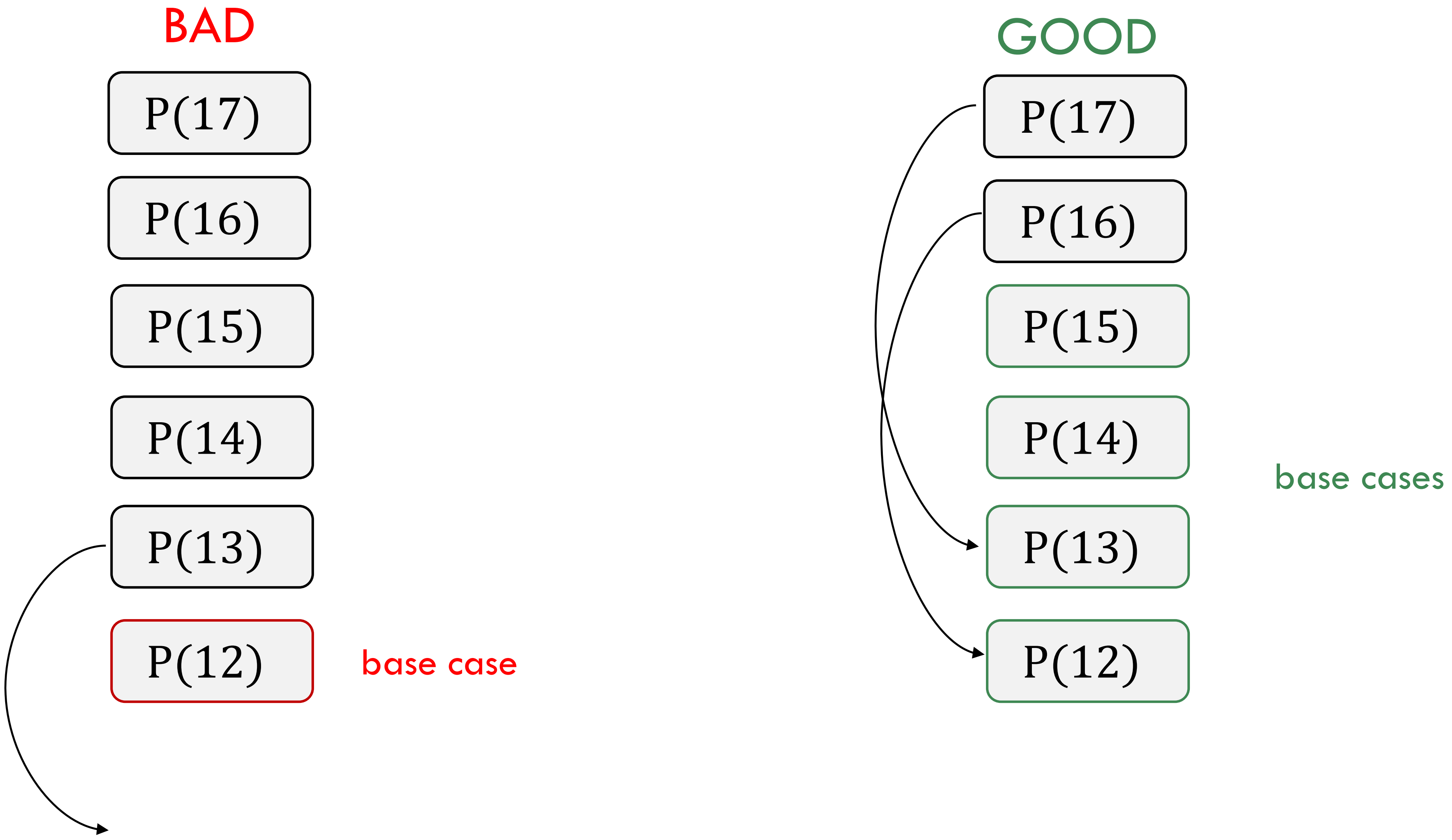
We don’t know $\mathrm{P}(13)$ holds.

When $k = 12$, and $k + 1 = 13$:

- Our IH assumes just $\mathrm{P}(12)$
- In the IS, we say since $\mathrm{P}(9)$ holds (going back to $k - 3$), then $\mathrm{P}(13)$ holds.
- But we don’t know anything about $\mathrm{P}(9)$! It might not even be true!

Lesson: If we go back $s$ steps in the IS, we need $s$ base cases.

# Tower Visualization



[Proof by Strong Induction]

Prove that for all $n \geq 12$, I can make $n$ ¢ worth of stamps.

1. Let P($n$) be "I can make n ¢ worth of stamps." We prove P(n) for all int $n \geq 12$ by strong induction.

2. Base Cases:

   12¢ : 3×4¢

   13¢ : 5¢, and 2×4¢

   14¢ : 2×5¢, and 4¢

   15¢ : 3× 5¢

   So P(12), P(13), P(14), P(15) hold.

3. IH: Suppose P(12) ∧ P(13) ∧ .... ∧ P(k) for an arbitrary integer $k \geq 15$.

4. IS: By IH we can make k-3 ¢. Adding a 4¢ stamp, we can make k+1 ¢.

5. Conclusion: Thus P(n) holds for all int $n \geq 12$ by strong induction.

# Strong Induction Lesson


Be careful about base cases!!

# Strong Induction Template

# Strong Induction Template

1. Define $\mathrm{P}(n)$. State that your proof is by strong induction on $n$.
2. Base Case: Show your base cases $\mathrm{P}(b_{\min}), \ldots, \mathrm{P}(b_{\max})$ are true.
3. Inductive Hypothesis: Suppose $\mathrm{P}(b_{\min}) \wedge \cdots \wedge \mathrm{P}(k)$ hold for an arbitrary integer $k \geq b_{\max}$ .
4. Inductive Step: Prove $\mathrm{P}(k+1)$ using the IH.
5. Conclusion: Conclude by saying $\mathrm{P}(n)$ holds for all integers $n \geq b_{\min}$ by strong induction.

# Practical Tip

- If you aren’t sure how many steps you’ll go back, leave space for the base cases.
- Do the IH / IS, and then fill in the base cases later.

# Strong Induction Example

Fibonacci Sequence

# Fibonacci Numbers

The Fibonacci Numbers are defined as follows:

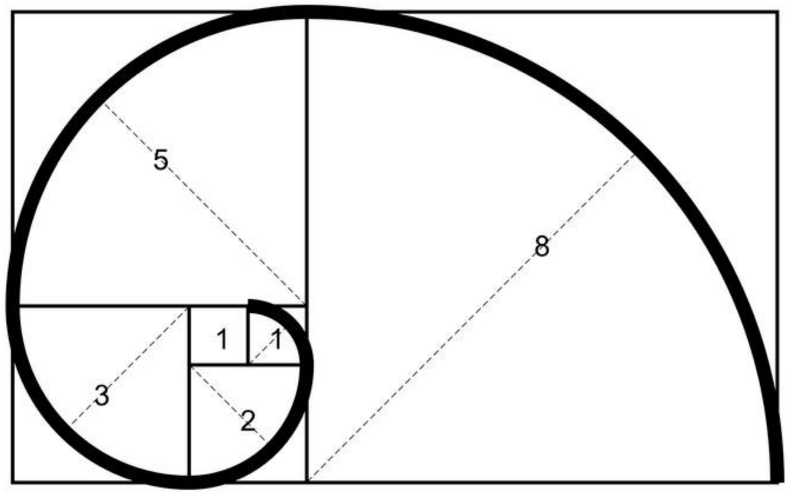
$f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

recursively defined function

0, 1, 1, 2, 3, 5, 8, 13, ...



# Fibonacci Numbers Claim

We claim that $f_n < 2^n$ for all $n \geq 0$.

$f_0 = 0$ $\quad$ $2^0 = 1$

$f_1 = 1$ $\quad$ $2^1 = 2$

$f_2 = 1$ $\quad$ $2^2 = 4$

$f_3 = 2$ $\quad$ $2^3 = 8$

$f_4 = 3$ $\quad$ $2^4 = 16$

$f_n = f_{n-1} + f_{n-2}$

We prove by strong induction!

# Prove that for all $n \in \mathbb{N}$, $f_n < 2^n$.

Definition:
$f_0 = 0, f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

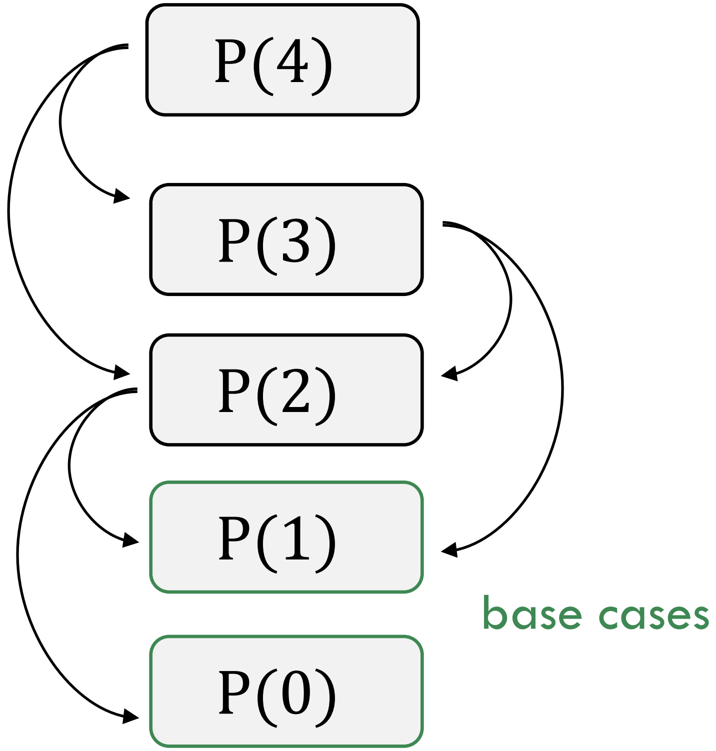
1. Let P($n$) be "$f_n < 2^n$". We prove P(n) for all $n \in \mathbb{N}$ by strong induction.
2. Base Case(s):
   P(0): Well $f_0 = 0$ and $2^0 = 1$ so $f_0 < 2^0$. So P(0) holds.
   P(1): Well $f_1 = 1$ and $2^1 = 2$ so $f_1 < 2^1$. So P(1) holds.
3. IH: Suppose $P(0) \wedge P(1) \wedge \ldots \wedge P(k)$ hold for an arbitrary integer $k \geq 1$.
4. IS: We aim to show P(k+1). Goal: $f_{k+1} < 2^{k+1}$

   Observe:

$$
\begin{aligned}
f_{k+1} &= f_k + f_{k-1} \\
&< 2^k + f_{k-1} && \text{by IH } (P(k)) \\
&< 2^k + 2^{k-1} && \text{by IH } (P(k-1)) \\
&\leq 2^k + 2^k && \text{since } 2^{k-1} = \tfrac{1}{2} \cdot 2^k \leq 2^k \\
&= 2^{k+1}
\end{aligned}
$$

   Thus P(k+1) holds.
5. Conclusion: Therefore P(n) holds for all $n \in \mathbb{N}$, by strong induction.

# Fibonacci Tower



# How many base cases?

- Always at least one base case.
- If you’re analyze a recursive function, at least one for each base case of the function.
- If you go back $s$ steps in the proof, at least $s$ base cases.