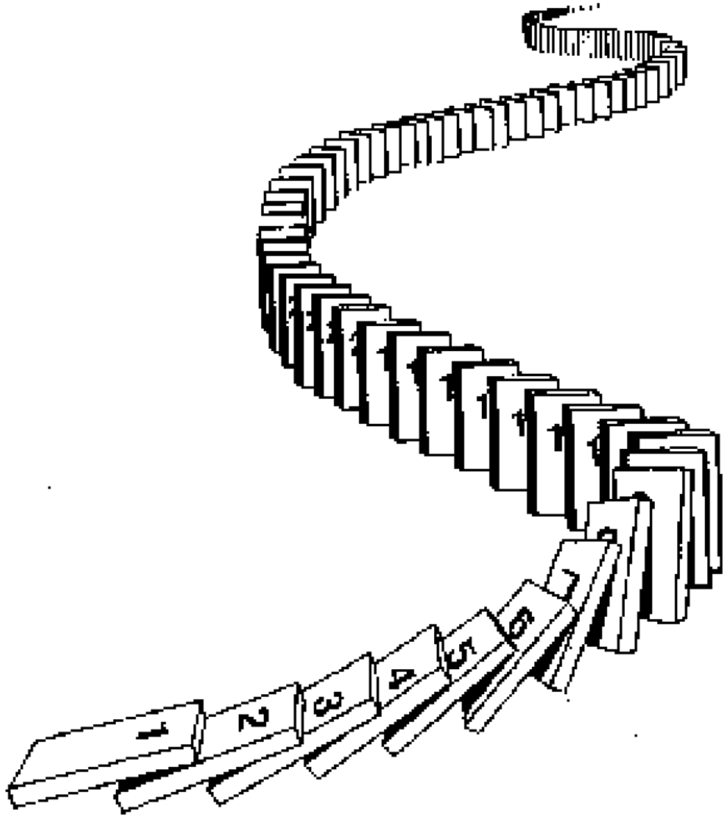
Warm-Up

Find an expression in terms of n for the sum $1 + 2 + 4 + \cdots + 2^n$

Hint:

- $n = 0$ $\quad 1 \quad = 1$
- $n = 1$ $\quad 1 + 2 \quad = 3$
- $n = 2$ $\quad 1 + 2 + 4 \quad = 7$
- $n = 3$ $\quad 1 + 2 + 4 + 8 \quad = 15$
- $n = 4$ $\quad 1 + 2 + 4 + 8 + 16 \quad = 31$



Induction

CSE 311: Foundations of Computing I
Lecture 13

Announcements

- HW4 due 11:59 pm tonight
- HW5 will be released tonight, due next Wednesday at 11:59 pm
  - We want to release feedback before your midterm next Friday (July 28th)
  - There will be 2 submission spots on Gradescope:

    HW5 (no late days) and HW5 (with late days)
  - Feedback before the midterm is only guaranteed if you don’t use late days
- Today is the last day of new content covered on the midterm
  - More details about the midterm in Friday’s lecture

Induction Motivation

Proof Strategies so Far

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Existence Proof

There are claims we cannot prove using these strategies!

Find an expression in n for the sum $1 + 2 + 4 + \cdots + 2^n$

- $n = 0$ $\quad 1 \quad = 1$
- $n = 1$ $\quad 1 + 2 \quad = 3$
- $n = 2$ $\quad 1 + 2 + 4 \quad = 7$
- $n = 3$ $\quad 1 + 2 + 4 + 8 \quad = 15$
- $n = 4$ $\quad 1 + 2 + 4 + 8 + 16 \quad = 31$

It *looks* like this sum is $2^{n+1} - 1$.

Find an expression in n for the sum $1 + 2 + 4 + \cdots + 2^n$

Claim: For all $n \in \mathbb{N}$, $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

In Logic: $\forall n \in \mathbb{N}\ (\mathrm{P}(n))$ where $\mathrm{P}(n)$ is the predicate "$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$".

How could we **prove** this claim?

- Naively? Prove $\mathrm{P}(0), \mathrm{P}(1), \mathrm{P}(2), \mathrm{P}(3), \ldots$ forever
- Direct Proof? Let $n \in \mathbb{N}$ be arbitrary. Then…

Idea!

Observe that:

$$\forall n\ \mathrm{P}(n) \equiv \mathrm{P}(0) \wedge \mathrm{P}(1) \wedge \mathrm{P}(2) \wedge \mathrm{P}(3) \ldots$$

$$\equiv \mathrm{P}(0) \wedge \big(\mathrm{P}(0) \rightarrow \mathrm{P}(1)\big) \wedge \big(\mathrm{P}(1) \rightarrow \mathrm{P}(2)\big) \wedge \cdots$$

$$\equiv \mathrm{P}(0) \wedge \forall k\ \big(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\big)$$

Induction:

- Check that $\mathrm{P}(0)$ holds
- Assume $\mathrm{P}(k)$ holds for arbitrary k and prove $\mathrm{P}(k+1)$ holds.

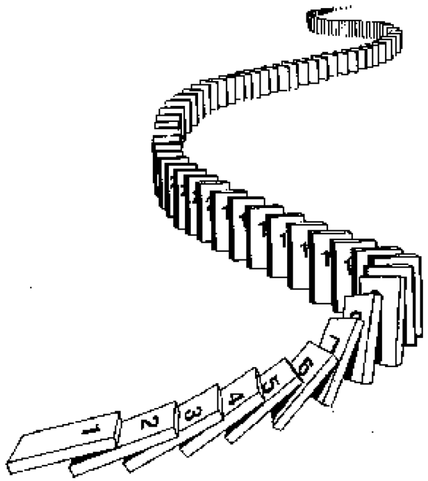
The Principle of Mathematical Induction

$$\mathrm{P}(0) \land \forall k\,\big(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\big)$$

Base Case
Prove $\mathrm{P}(0)$ holds.

Inductive Hypothesis
Let $k \geq 0$ be an arbitrary integer. Suppose $\mathrm{P}(k)$ holds.

Inductive Step
Prove that $\mathrm{P}(k+1)$ holds (using $\mathrm{P}(k)$)



Prove $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

To prove $\forall n \; \mathrm{P}(n)$, prove
$\mathrm{P}(0) \land \forall k \left(\mathrm{P}(k) \to \mathrm{P}(k+1)\right)$

1. Let $\mathrm{P}(n)$ be "$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$". We prove $\mathrm{P}(n)$ for all $n \in \mathbb{N}$ by induction.
2. Base Case: The LHS evaluates to 1. The RHS evaluates to $2^{0+1} - 1 = 2^1 - 1 = 1$. Since $1 = 1$, the base case holds.
3. Inductive Hypothesis: Suppose $\mathrm{P}(k)$ holds for an arbitrary integer $k \geq 0$. That is, $1 + 2 + 4 + \cdots + 2^k = 2^{k+1} - 1$.
4. Inductive Step: We aim to show $\mathrm{P}(k+1)$. Observe that:

$$\begin{aligned} 1 + 2 + 4 + \cdots + 2^{k+1} &= 1 + 2 + 4 + \cdots + 2^k + 2^{k+1} \\ &= 2^{k+1} - 1 + 2^{k+1} && \text{By the IH} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

   So $\mathrm{P}(k+1)$ holds.
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all $n \in \mathbb{N}$ by induction.

Induction Template

Induction Template

1. Define $\mathrm{P}(n)$. State that your proof is by induction on n.
2. Base Case: Show $\mathrm{P}(b)$ is true for your base case b.
3. Inductive Hypothesis: Suppose $\mathrm{P}(k)$ holds for an arbitrary integer $k \geq b$.
4. Inductive Step: Prove $\mathrm{P}(k+1)$ (using the Inductive Hypothesis).
5. Conclusion: Conclude by saying $\mathrm{P}(n)$ holds for all integers $n \geq b$ by induction.

Induction Examples

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Examples

$n = 3$ Sum: $1 + 2 + 3 = 6$

Formula: $\frac{3(3+1)}{2} = \frac{3 \cdot 4}{2} = 6$

$n = 5$ Sum: $1 + 2 + 3 + 4 + 5 = 15$

Formula: $\frac{5(5+1)}{2} = \frac{5 \cdot 6}{2} = 15$

Carl Friedrich Gauss
(1777-1855)

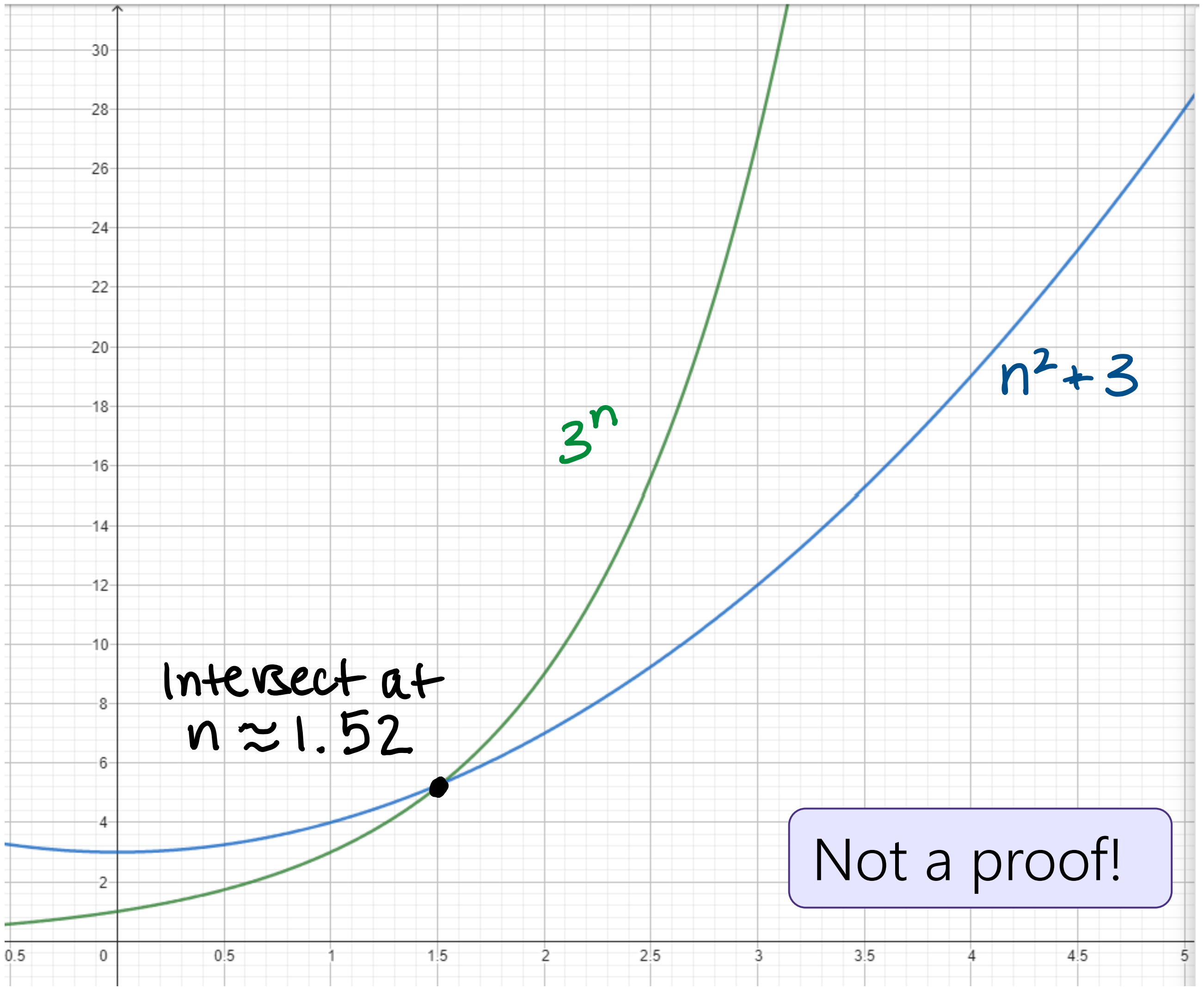
Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

1. Let $\mathrm{P}(n)$ be "$1+2+3+\cdots+n=\frac{n(n+1)}{2}$". We prove $\mathrm{P}(n)$ for all **integers $n \geq 1$** by induction.
2. Base Case: The LHS evaluates to 1. The RHS evaluates to $\frac{1(1+1)}{2}=1$. Since $1=1$, the base case holds.
3. Inductive Hypothesis: Suppose $\mathrm{P}(k)$ holds for an arbitrary integer $k \geq 1$. That is, $1+2+3+\cdots+k=\frac{k(k+1)}{2}$.
4. Inductive Step: We aim to show $\mathrm{P}(k+1)$. Observe that:

$$
\begin{aligned}
1+2+3+\cdots+(k+1) &= 1+2+3+\cdots+k+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) && \text{By the IH} \\
&= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}
\end{aligned}
$$

   So $\mathrm{P}(k+1)$ holds.
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 1$ by induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.



Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

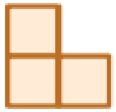
1. Let $\mathrm{P}(n)$ be "$3^n \geq n^2 + 3$". We prove $\mathrm{P}(n)$ for all **integers $n \geq 2$** by induction.
2. Base Case ($n = 2$): The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.
3. Inductive Hypothesis: Suppose $\mathrm{P}(k)$ holds for an arbitrary integer $k \geq 2$. That is, $3^k \geq k^2 + 3$.
4. Inductive Step: We aim to show $\mathrm{P}(k+1)$. Observe that:

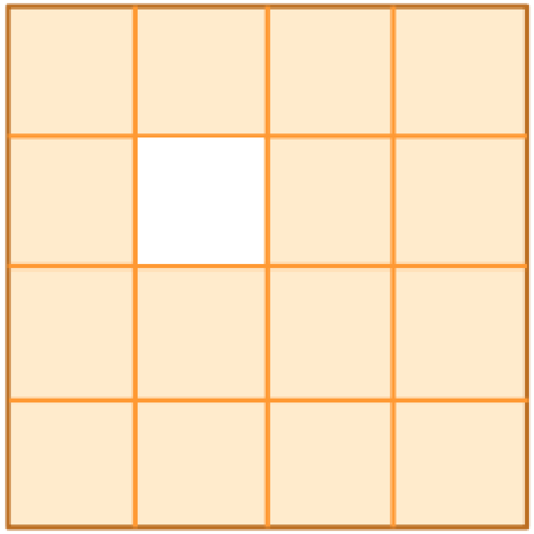
   $$\begin{aligned} 3^{k+1} &= 3 \cdot 3^k \\ &\geq 3 \cdot (k^2 + 3) && \text{By the IH} \\ &= 3k^2 + 9 \\ &= k^2 + 2k^2 + 9 \\ &\geq k^2 + 2k + 9 && \text{Since } k \geq 1 \text{, so } 2k^2 \geq 2k \\ &\geq (k+1)^2 + 3 \end{aligned}$$

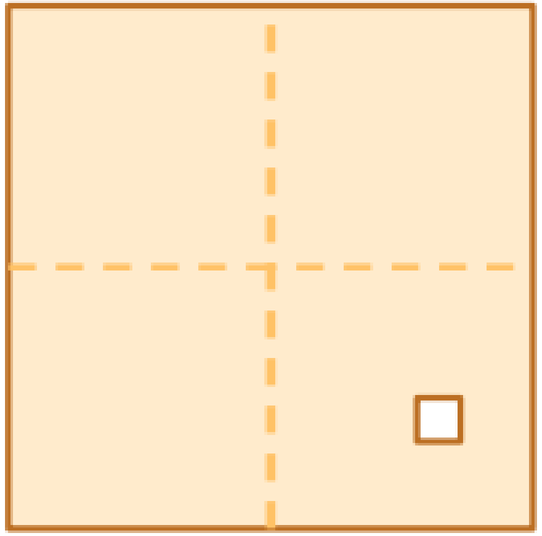
   So $\mathrm{P}(k+1)$ holds.
5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 2$ by induction.

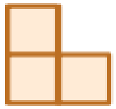
Checkerboard Tiling

Imagine a $2^n \times 2^n$ checkerboard with a single square removed.

Can you tile the board with  pieces? You may rotate and flip the pieces around.

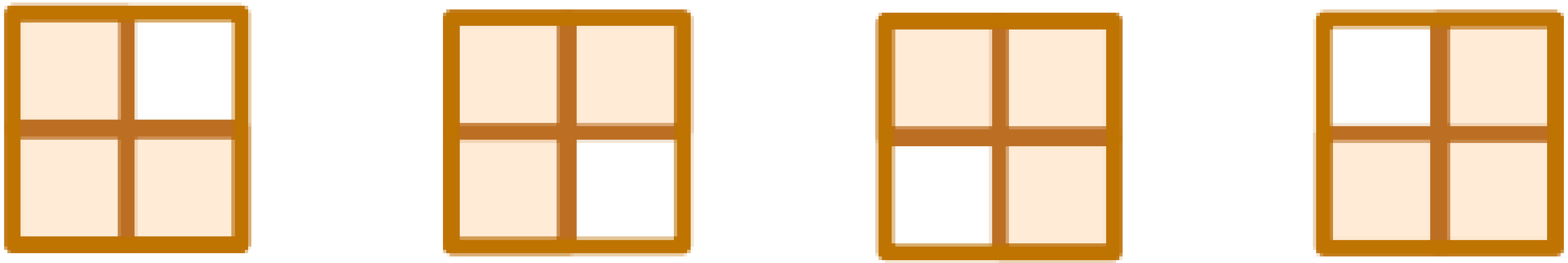




Claim: All $2^n \times 2^n$ boards with one square removed can be tiled with  pieces

Checkerboard Tiling: Base Case

Consider all 2×2 boards with one piece missing.



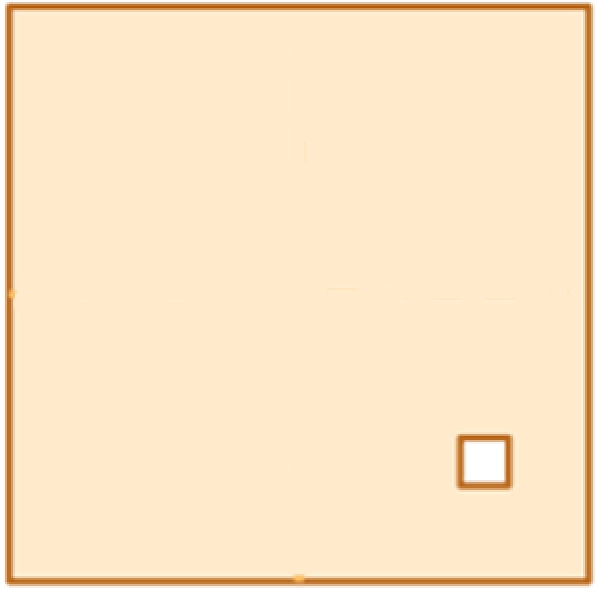
We can definitely tile these with one piece!

Checkerboard Tiling: Inductive Hypothesis

Assume you could tile any $2^k \times 2^k$ board with one piece missing.

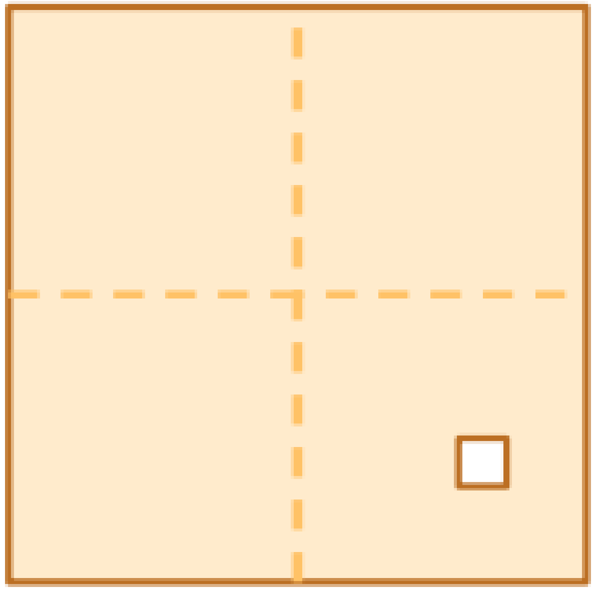
Checkerboard Tiling: Inductive Step

Now consider a $2^{k+1} \times 2^{k+1}$ board with one piece missing.



Checkerboard Tiling: Inductive Step

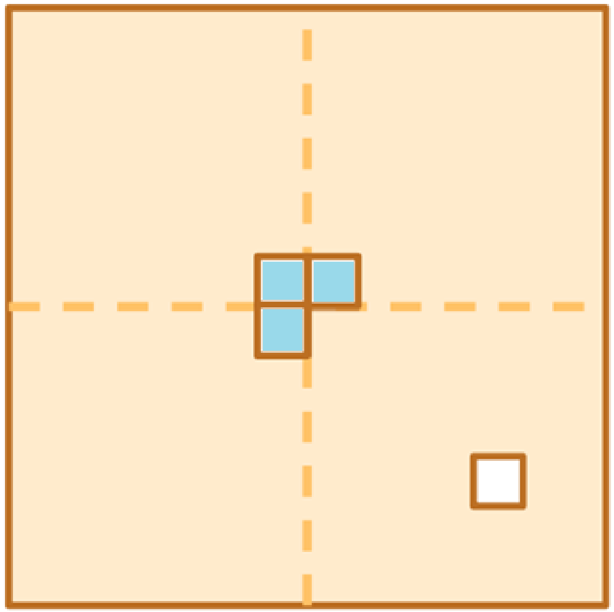
Now consider a $2^{k+1} \times 2^{k+1}$ board with one piece missing.



Divide the board into four quadrants of dimension $2^k \times 2^k$.

Checkerboard Tiling: Inductive Step

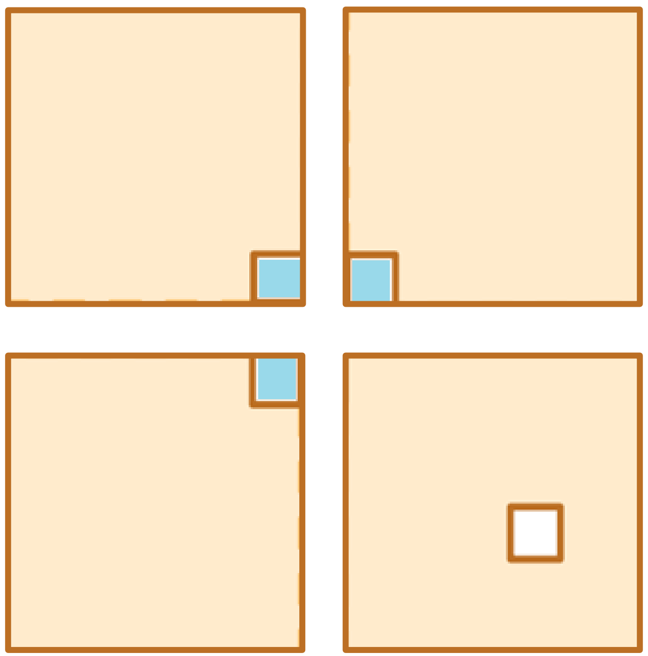
Now consider a $2^{k+1} \times 2^{k+1}$ board with one piece missing.



Place a single piece to occupy the three quadrants that aren't missing a piece.

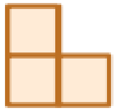
Checkerboard Tiling: Inductive Step

Now consider a $2^{k+1} \times 2^{k+1}$ board with one piece missing.



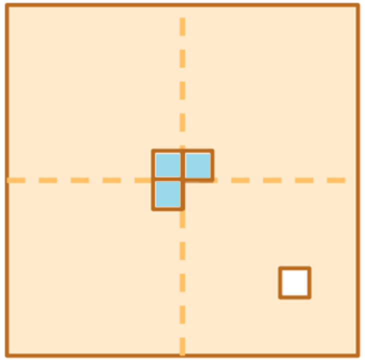
Each quadrant is now a $2^k \times 2^k$ board with one piece missing.
We can tile each of these by the IH.

Checkerboard Tiling

1. Let $\mathrm{P}(n)$ be "all $2^n \times 2^n$ boards with one square removed can be tiled with  pieces." We prove $\mathrm{P}(n)$ for all integers $n \geq 1$ by induction.
2. Base Case ($n = 1$): Observe that we can tile all 2×2 checkerboards:

   So the base case holds.
3. Inductive Hypothesis: Suppose $\mathrm{P}(k)$ holds for an arbitrary integer $k \geq 1$. That is, assume we can tile all $2^k \times 2^k$ checkerboards with one piece missing.
4. Inductive Step: We aim to show $\mathrm{P}(k+1)$. Consider an arbitrary $2^{k+1} \times 2^{k+1}$ checkerboard. We can divide the board into four quadrants, with one piece missing in one quadrant. Now place a single piece to occupy the three quadrants that aren't missing a piece. We now have four $2^k \times 2^k$ quadrants that are effectively each missing a piece. By the IH, we can tile each quadrant. Thus we can tile the entire checkerboard. So $\mathrm{P}(k+1)$ holds.

5. Conclusion: Thus $\mathrm{P}(n)$ holds for all integers $n \geq 1$ by induction.