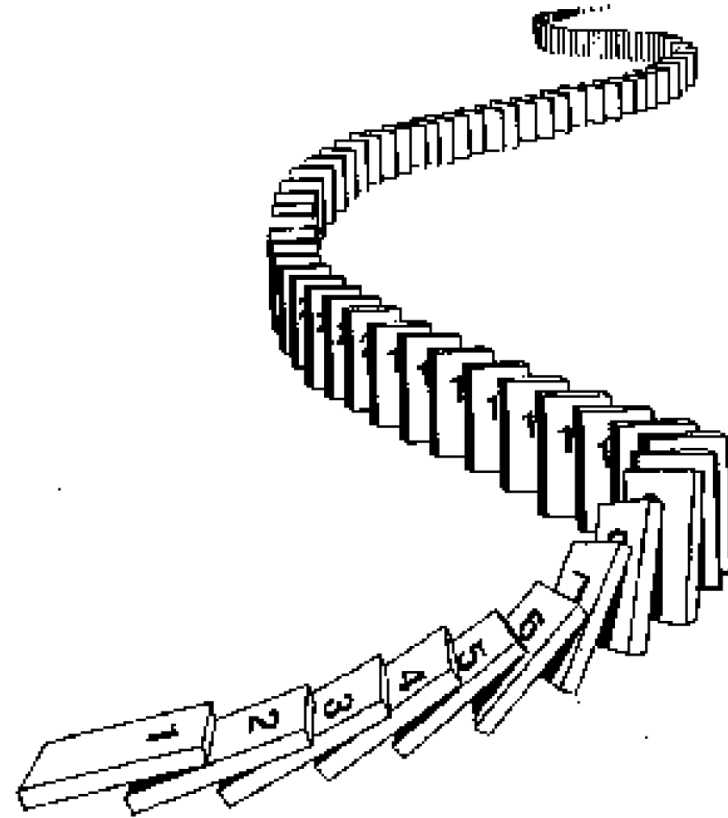


# Warm-Up

Find an expression in terms of  $n$  for the sum  $1 + 2 + 4 + \dots + 2^n$

Hint:

- $n = 0$                      $1$                      $= 1$
- $n = 1$                      $1 + 2$                      $= 3$
- $n = 2$                      $1 + 2 + 4$                      $= 7$
- $n = 3$                      $1 + 2 + 4 + 8$                      $= 15$
- $n = 4$                      $1 + 2 + 4 + 8 + 16$                      $= 31$

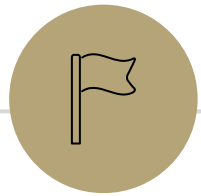


# Induction

CSE 311: Foundations of  
Computing I  
Lecture 13

# Announcements

- HW4 due 11:59 pm tonight
- HW5 will be released tonight, due next Wednesday at 11:59 pm
  - We want to release feedback before your midterm next Friday (July 28<sup>th</sup>)
  - There will be 2 submission spots on Gradescope:  
HW5 (no late days) and HW5 (with late days)
  - Feedback before the midterm is only guaranteed if you don't use late days
- Today is the last day of new content covered on the midterm
  - More details about the midterm in Friday's lecture



# Induction Motivation

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# Proof Strategies so Far

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Existence Proof

There are claims we cannot prove using these strategies!

Find an expression in  $n$  for the sum  $1 + 2 + 4 + \dots + 2^n$

- $n = 0$                      $1$                      $= 1$
- $n = 1$                      $1 + 2$                      $= 3$
- $n = 2$                      $1 + 2 + 4$                      $= 7$
- $n = 3$                      $1 + 2 + 4 + 8$                      $= 15$
- $n = 4$                      $1 + 2 + 4 + 8 + 16$                      $= 31$

It *looks* like this sum is  $2^{n+1} - 1$ .

Find an expression in  $n$  for the sum  $1 + 2 + 4 + \dots + 2^n$

Claim: \_\_\_\_\_

In Logic: \_\_\_\_\_

---

How could we **prove** this claim?

Idea!

Observe that:

Induction:

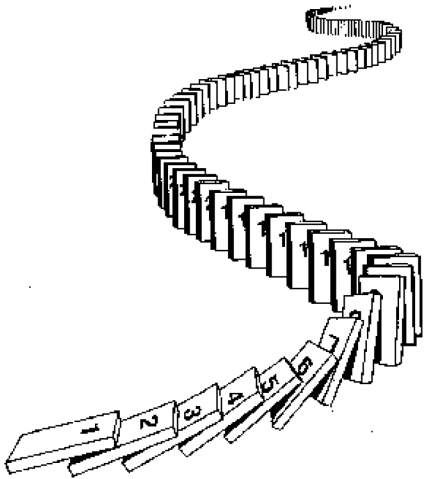
# The Principle of Mathematical Induction

$$P(0) \wedge \forall k (P(k) \rightarrow P(k + 1))$$

Base Case  
Prove  $P(0)$  holds.

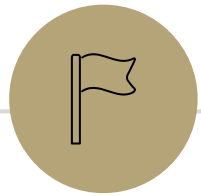
Inductive Hypothesis  
Let  $k \geq 0$  be an  
arbitrary integer.  
Suppose  $P(k)$  holds.

Inductive Step  
Prove that  $P(k + 1)$   
holds (using  $P(k)$ )



Prove  $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$  for all  $n \in \mathbb{N}$ .

To prove  $\forall n P(n)$ , prove  
 $P(0) \wedge \forall k (P(k) \rightarrow P(k + 1))$



# Induction Template



# Induction Template

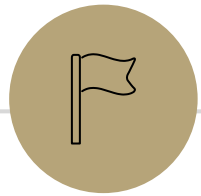
1.

2.

3.

4.

5.



# Induction Examples

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Prove that the sum of the first  $n$  positive integers is  $\frac{n(n+1)}{2}$ .

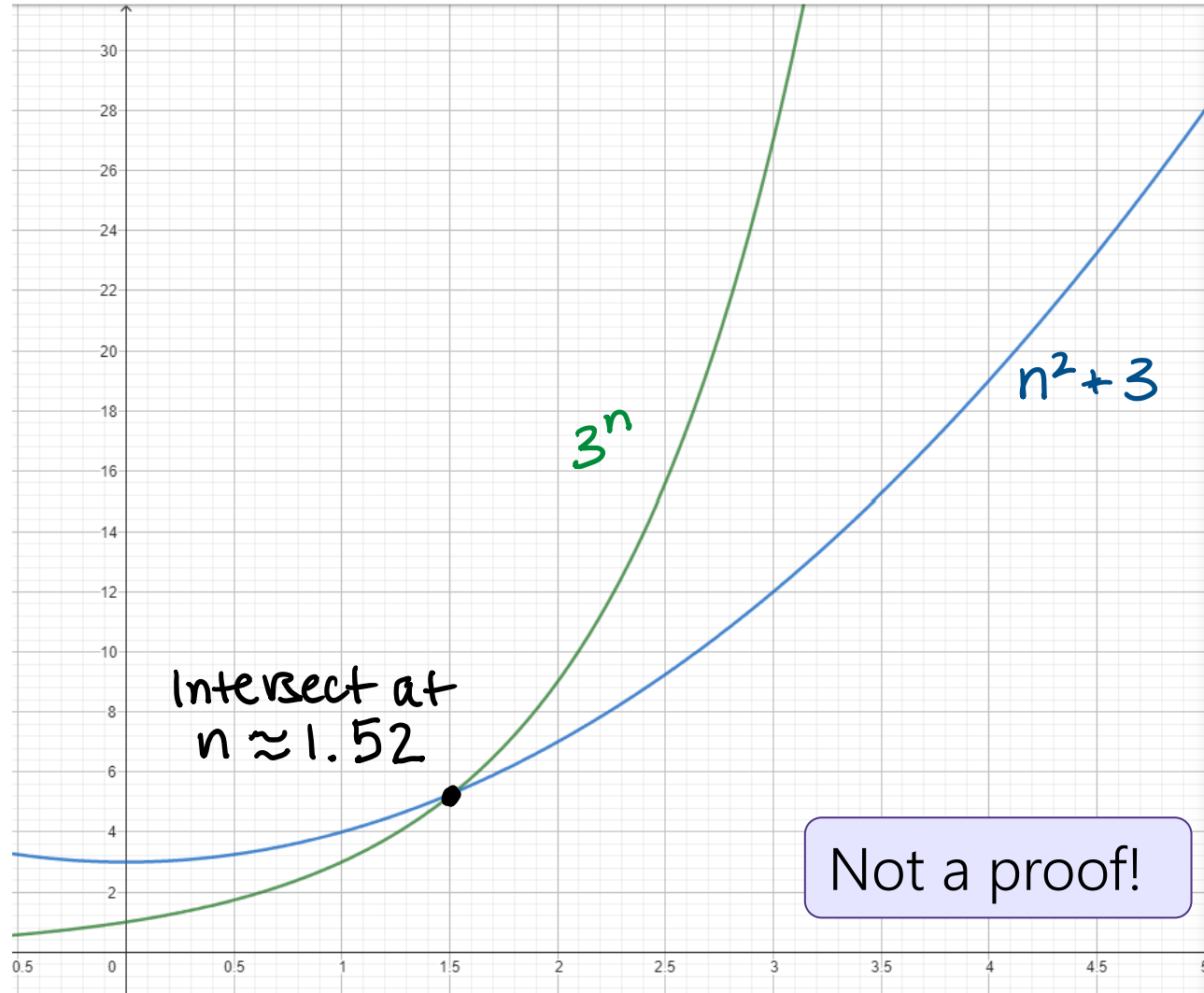
Examples



Carl Friedrich Gauss  
(1777-1855)

Prove that the sum of the first  $n$  positive integers is  $\frac{n(n+1)}{2}$ .

Prove that  $3^n \geq n^2 + 3$  for all integers  $n \geq 2$ .

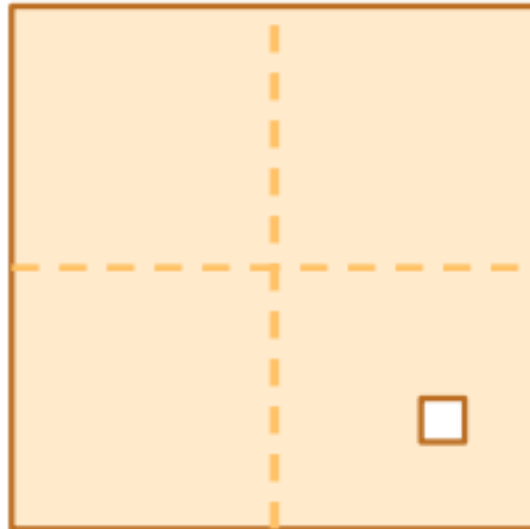
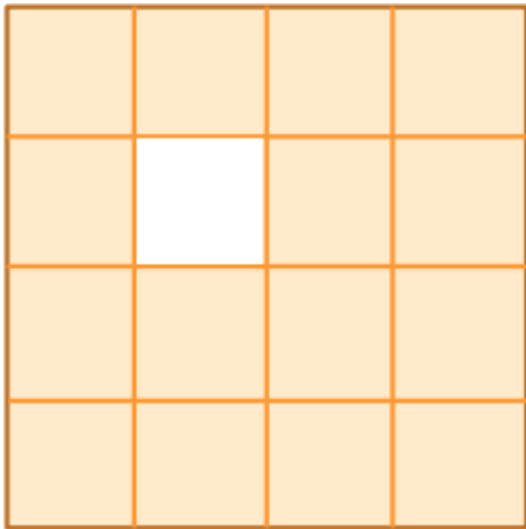



Prove that  $3^n \geq n^2 + 3$  for all integers  $n \geq 2$ .

# Checkerboard Tiling

Imagine a  $2^n \times 2^n$  checkerboard with a single square removed.

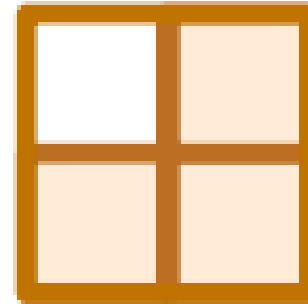
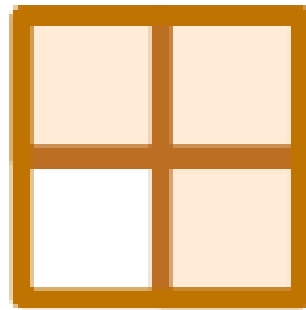
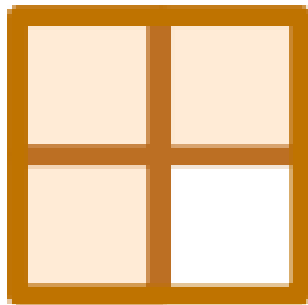
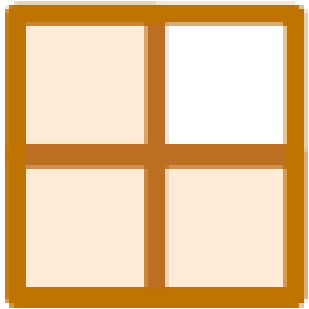
Can you tile the board with  pieces? You may rotate and flip the pieces around.



Claim: All  $2^n \times 2^n$  boards with one square removed can be tiled with  pieces

# Checkerboard Tiling: Base Case

Consider all  $2 \times 2$  boards with one piece missing.



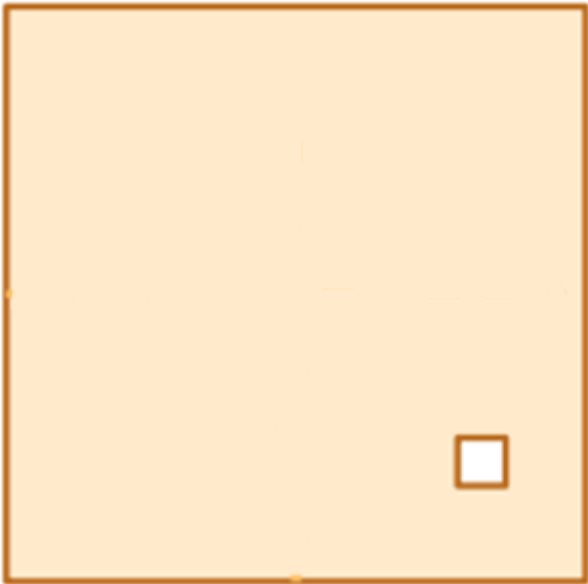
We can definitely tile these with one piece!

# Checkerboard Tiling: Inductive Hypothesis

**Assume** you could tile any  $2^k \times 2^k$  board with one piece missing.

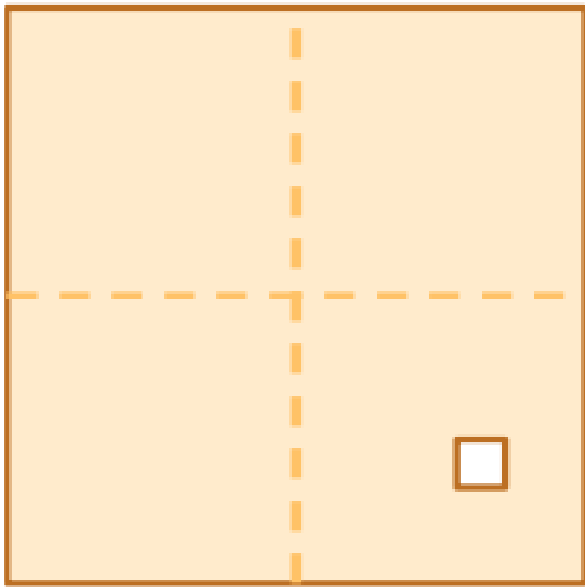
# Checkerboard Tiling: Inductive Step

Now consider a  $2^{k+1} \times 2^{k+1}$  board with one piece missing.



# Checkerboard Tiling: Inductive Step

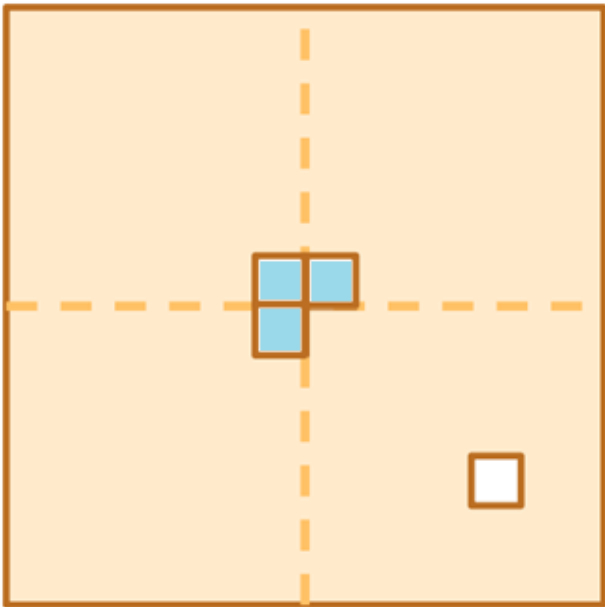
Now consider a  $2^{k+1} \times 2^{k+1}$  board with one piece missing.



Divide the board into four quadrants of dimension  $2^k \times 2^k$ .

# Checkerboard Tiling: Inductive Step

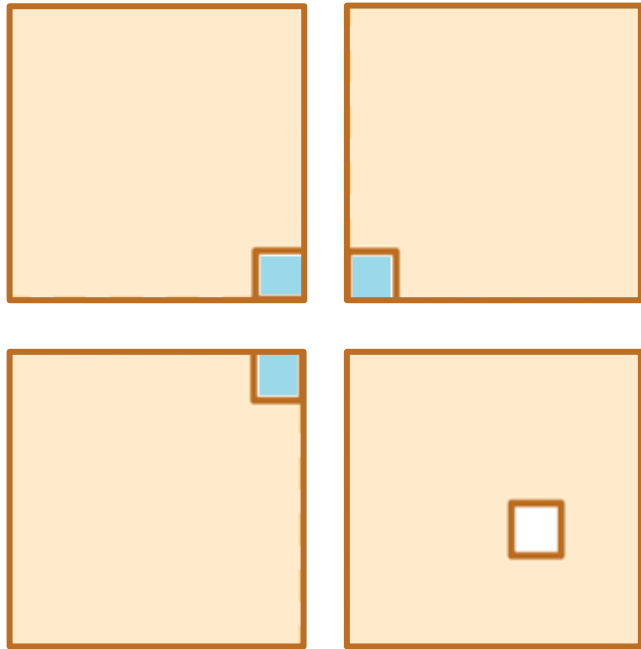
Now consider a  $2^{k+1} \times 2^{k+1}$  board with one piece missing.



Place a single piece to occupy the three quadrants that aren't missing a piece.


# Checkerboard Tiling: Inductive Step

Now consider a  $2^{k+1} \times 2^{k+1}$  board with one piece missing.



Each quadrant is now a  $2^k \times 2^k$  board with one piece missing. We can tile each of these by the IH.

# Checkerboard Tiling

1. Let  $P(n)$  be "all  $2^n \times 2^n$  boards with one square removed can be tiled with  pieces." We prove  $P(n)$  for all integers  $n \geq 1$  by induction.

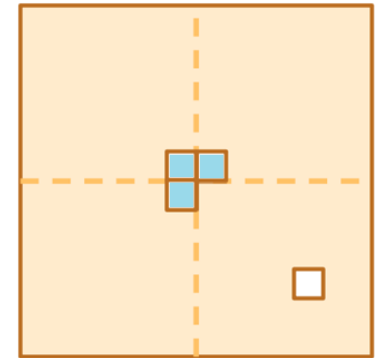
2. Base Case ( $n = 1$ ): Observe that we can tile all  $2 \times 2$  checkerboards:



So the base case holds.

3. Inductive Hypothesis: Suppose  $P(k)$  holds for an arbitrary integer  $k \geq 1$ . That is, assume we can tile all  $2^k \times 2^k$  checkerboards with one piece missing.

4. Inductive Step: We aim to show  $P(k + 1)$ . Consider an arbitrary  $2^{k+1} \times 2^{k+1}$  checkerboard. We can divide the board into four quadrants, with one piece missing in one quadrant. Now place a single piece to occupy the three quadrants that aren't missing a piece. We now have four  $2^k \times 2^k$  quadrants that are effectively each missing a piece. By the IH, we can tile each quadrant. Thus we can tile the entire checkerboard. So  $P(k + 1)$  holds.



5. Conclusion: Thus  $P(n)$  holds for all integers  $n \geq 1$  by induction.