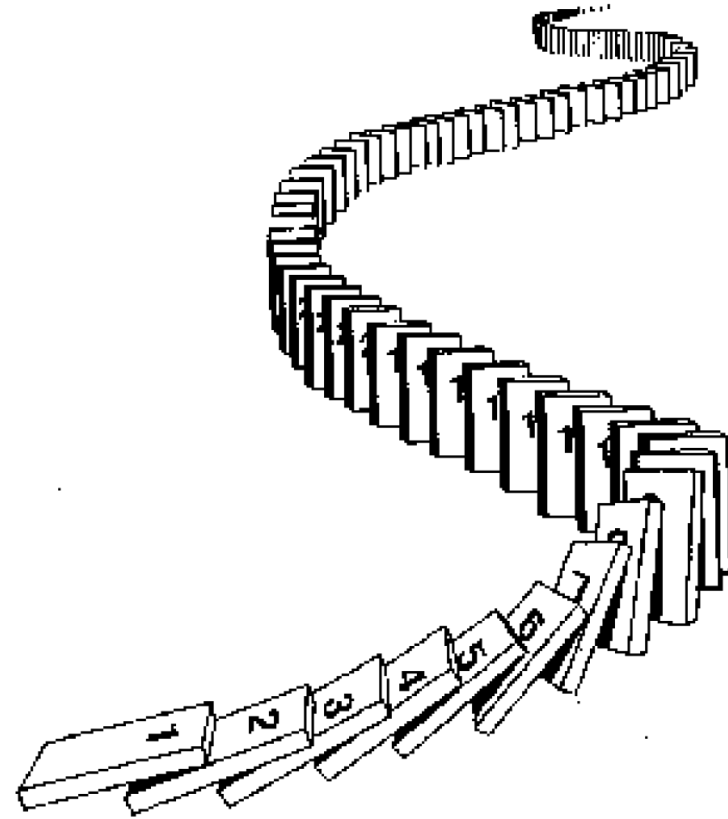


Warm-Up

Find an expression in terms of n for the sum $1 + 2 + 4 + \dots + 2^n$

Hint:

- $n = 0$ 1 $= 1$
- $n = 1$ $1 + 2$ $= 3$
- $n = 2$ $1 + 2 + 4$ $= 7$
- $n = 3$ $1 + 2 + 4 + 8$ $= 15$
- $n = 4$ $1 + 2 + 4 + 8 + 16$ $= 31$

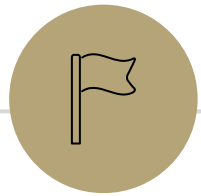


Induction

CSE 311: Foundations of
Computing I
Lecture 13

Announcements

- HW4 due 11:59 pm tonight
- HW5 will be released tonight, due next Wednesday at 11:59 pm
 - We want to release feedback before your midterm next Friday (July 28th)
 - There will be 2 submission spots on Gradescope:
HW5 (no late days) and HW5 (with late days)
 - Feedback before the midterm is only guaranteed if you don't use late days
- Today is the last day of new content covered on the midterm
 - More details about the midterm in Friday's lecture



Induction Motivation

Proof Strategies so Far

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Existence Proof

There are claims we cannot prove using these strategies!

Find an expression in n for the sum $1 + 2 + 4 + \dots + 2^n$

- $n = 0$ 1 = 1 2 $\sum_{i=0}^n 2^i$
 - $n = 1$ 1 + 2 = 3 4
 - $n = 2$ 1 + 2 + 4 = 7 8
 - $n = 3$ 1 + 2 + 4 + 8 = 15 16
 - $n = \textcircled{4}$ 1 + 2 + 4 + 8 + 16 = 31 32
- $2^{n+1} - 1$
- $2^1 - 1 = 1 \checkmark$
- $2^5 - 1 = 31 \checkmark$

It *looks* like this sum is $2^{n+1} - 1$.

Find an expression in n for the sum $1 + 2 + 4 + \dots + 2^n$

Claim: For all $n \in \mathbb{N}$, $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$

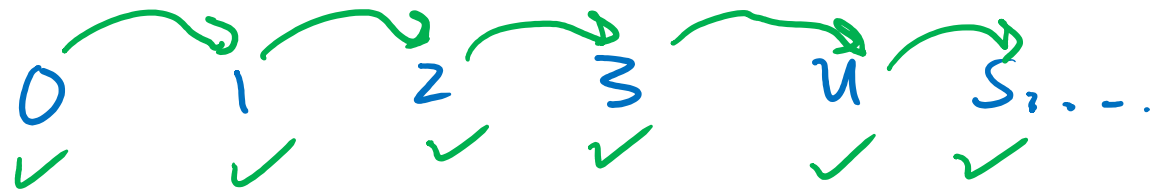
In Logic: $\forall n \in \mathbb{N} (P(n))$ where $P(n)$ is the predicate

" $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$ "

How could we **prove** this claim?

- Naïvely: $P(0), P(1), P(2), P(3), \dots$
- Direct Proof: Let $n \in \mathbb{N}$ be arbitrary.
 $1 + 2 + 4 + \dots + 2^n = \dots = \underline{2^{n+1} - 1}$

Idea!



Observe that:

$$\begin{aligned}\forall n P(n) &\equiv P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \dots \\ &\equiv \underline{P(0)} \wedge (P(0) \rightarrow P(1)) \wedge (P(1) \rightarrow P(2)) \wedge \dots \\ &\equiv P(0) \wedge \forall k (P(k) \rightarrow P(k+1))\end{aligned}$$

Induction:

- Check that $P(0)$ holds
- Let k be an arbitrary $\text{int} \geq 0$. Assume $P(k)$. Prove $P(k+1)$.

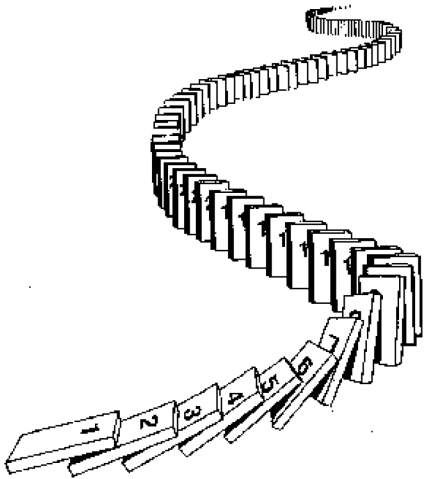
The Principle of Mathematical Induction

$$\forall n P(n) \equiv P(0) \wedge \forall k (P(k) \rightarrow P(k+1))$$

Base Case
Prove $P(0)$ holds.

Inductive Hypothesis
Let $k \geq 0$ be an
arbitrary integer.
Suppose $P(k)$ holds.

Inductive Step
Prove that $P(k+1)$
holds (using $P(k)$)



Prove $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

To prove $\forall n P(n)$, prove
 $P(0) \wedge \forall k (P(k) \rightarrow P(k+1))$

1. Let $P(n)$ be " $1+2+4+\dots+2^n = 2^{n+1}-1$." We prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

2. ~~Base case~~ ($n=0$): The LHS evaluates to 1. The RHS evaluates to $2^{0+1}-1 = 2-1=1$. Since $1=1$, the base case holds.

3. IH: Suppose $P(k)$ holds for an arbitrary integer $k \geq 0$. That is

$$1 + 2 + 4 + \dots + 2^k = 2^{k+1} - 1.$$

4. IS: Goal $P(k+1)$ $1 + 2 + 4 + \dots + 2^{k+1} = 2^{k+2} - 1$

$$1 + 2 + 4 + \dots + 2^{k+1} = 1 + 2 + 4 + \dots + 2^k + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1}$$

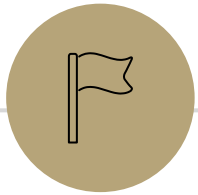
$$= 2 \cdot 2^{k+1} - 1$$

$$= \underline{2^{k+2} - 1}$$

By IH

Thus $P(k+1)$ holds

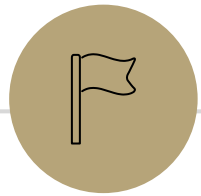
5. Conclusion: Thus $P(n)$ holds for all $n \in \mathbb{N}$ by induction.



Induction Template

Induction Template

1. Define $P(n)$. State that the proof is by induction.
2. Base case: Show $P(b)$ is true for base case b .
3. Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary integer $k \geq b$.
4. Inductive Step: Prove $P(k+1)$ - use IH.
5. Conclusion - state that $P(n)$ holds for all integers $n \geq b$ by induction.



Induction Examples

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Examples

$$n=3 \quad 1+2+3 = 6$$

$$\text{formula: } \frac{3 \cdot 4}{2} = \frac{12}{2} = 6 \checkmark$$

$$n=5 \quad 1+2+3+4+5 = 15$$

$$\text{formula: } \frac{5 \cdot 6}{2} = \frac{30}{2} = 15 \checkmark$$



Carl Friedrich Gauss
(1777-1855)

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

1. Let $P(n)$ be " $1+2+3+\dots+n = \frac{n(n+1)}{2}$ ". We prove $P(n)$ for all integers $n \geq 1$ by induction.

2. Base case ($n=1$): LHS evaluates to 1. RHS evaluates to $\frac{1 \cdot 2}{2} = 1$. Since $1=1$, the base case holds.

3. IH: suppose $P(k)$ holds for an arbitrary integer $k \geq 1$. That is, $1+2+\dots+k = \frac{k(k+1)}{2}$.

4. IS: $1+2+\dots+(k+1) = \frac{(k+1)(k+2)}{2}$

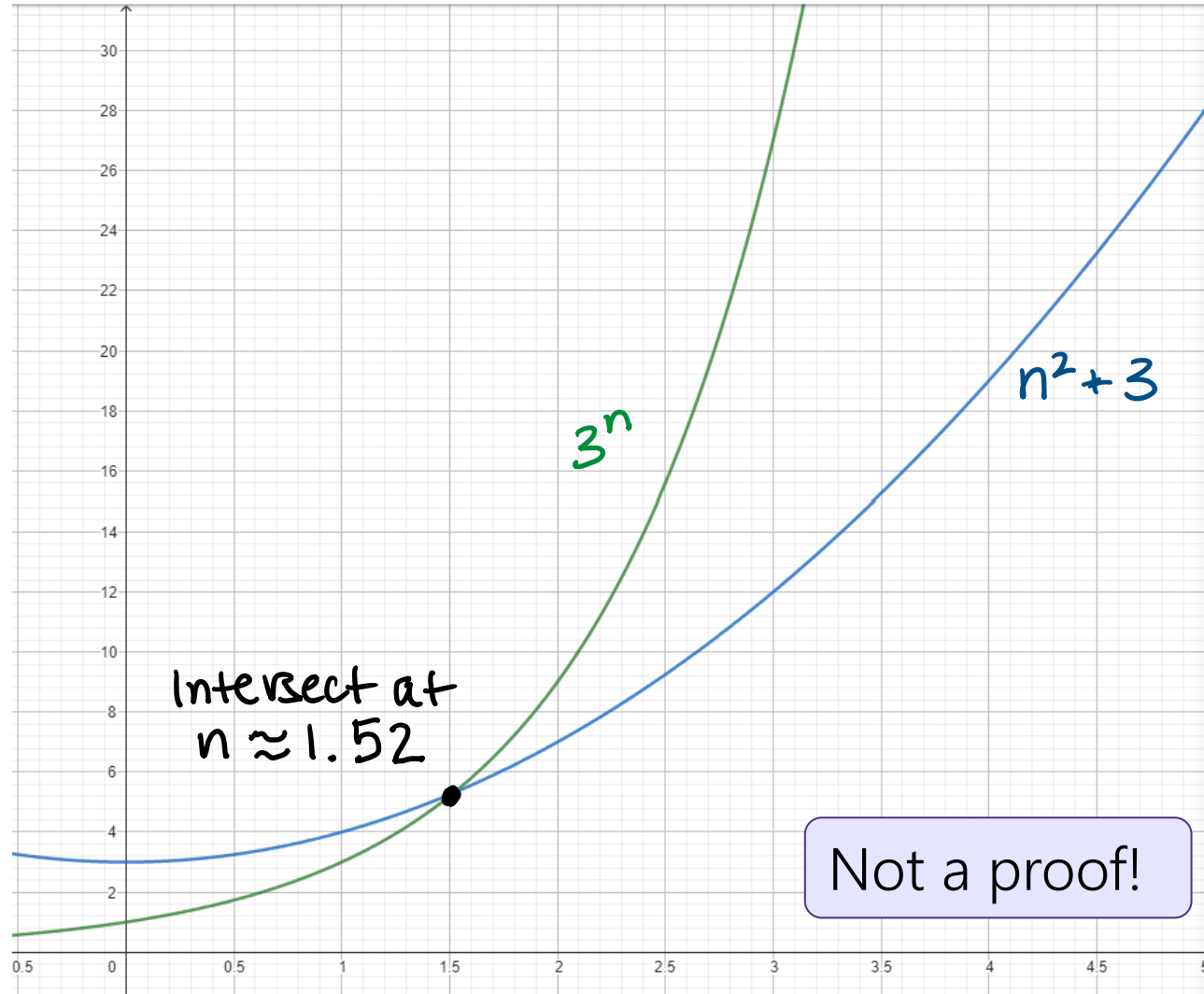
$$\begin{aligned} \text{Observe: } 1+2+\dots+(k+1) &= 1+2+\dots+k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \end{aligned}$$

by IH

$$\text{So } P(k+1) \text{ holds} \quad = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

5. Conclusion: Thus $P(n)$ holds for all integers $n \geq 1$ by induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.



Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

1. Let $P(n)$ be " $3^n \geq n^2 + 3$ ". We prove $P(n)$ for all int $n \geq 2$ by induction.

2. Base case ($n=2$): LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, base case holds.

3. IH: suppose $P(k)$ holds for an arbitrary integer $k \geq 2$. i.e.
 $3^k \geq k^2 + 3$.

4. IS: Goal: $P(k+1): 3^{k+1} \geq (k+1)^2 + 3$

$$3^{k+1} = 3 \cdot 3^k$$

$$\geq 3 \cdot (k^2 + 3) \quad \text{by IH}$$

$$= 3k^2 + 9$$

$$= k^2 + 2k^2 + 9$$

$$\geq k^2 + 2k + 9 \quad \text{since } k \geq 1, k^2 \geq k, \text{ so } 2k^2 \geq 2k$$

$$\geq k^2 + 2k + 4$$

$$\geq (k+1)^2 + 3$$

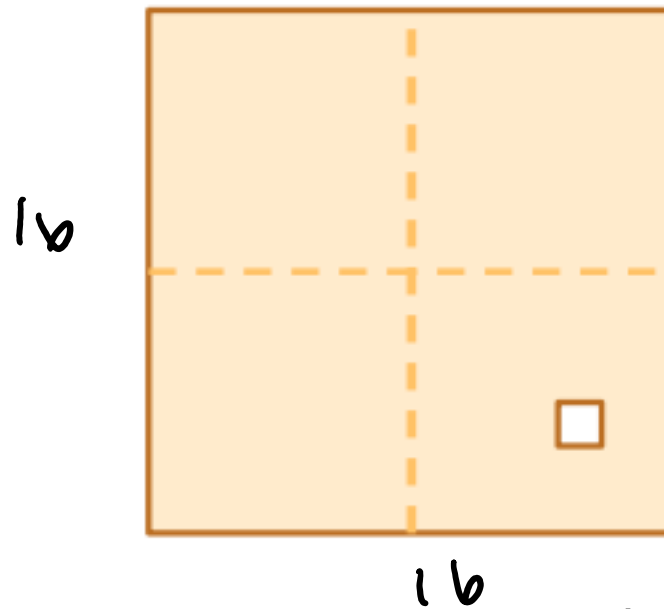
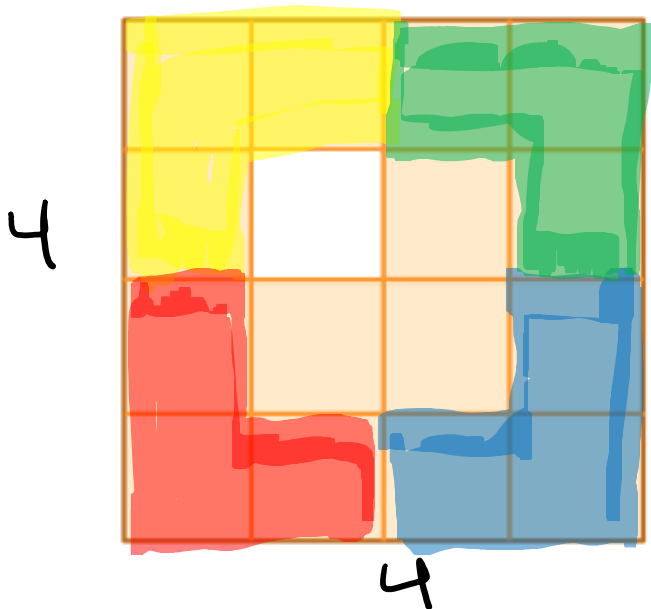
since $9 \geq 4$
Thus $P(k+1)$ holds


S. Thus $P(n)$ holds for all integers $n \geq 2$ by induction.

Checkerboard Tiling

Imagine a $2^n \times 2^n$ checkerboard with a single square removed.

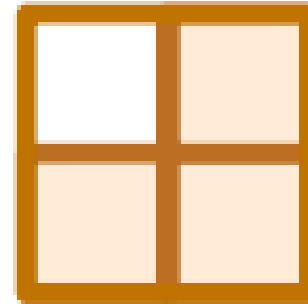
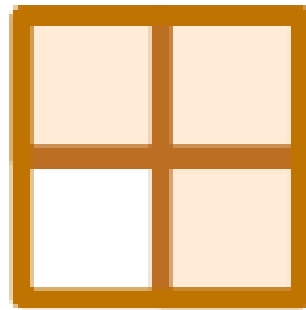
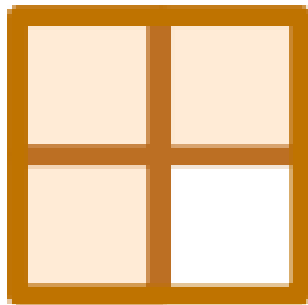
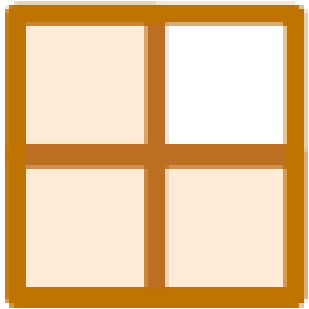
Can you tile the board with  pieces? You may rotate and flip the pieces around.



Claim: All $2^n \times 2^n$ boards with one square removed can be tiled with  pieces

Checkerboard Tiling: Base Case

Consider all 2×2 boards with one piece missing.



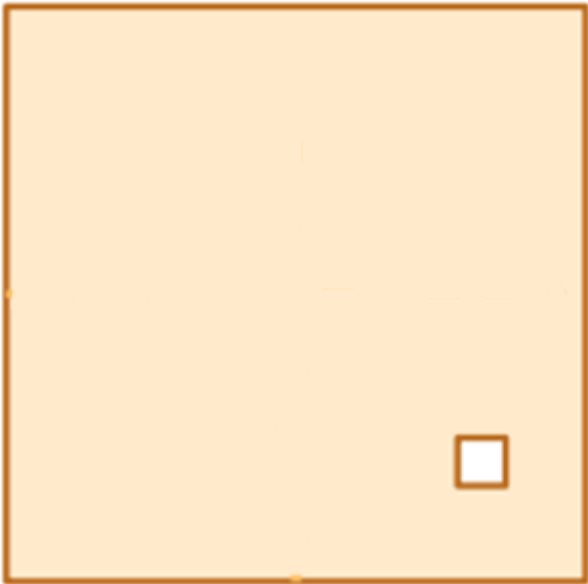
We can definitely tile these with one piece!

Checkerboard Tiling: Inductive Hypothesis

Assume you could tile any $2^k \times 2^k$ board with one piece missing.

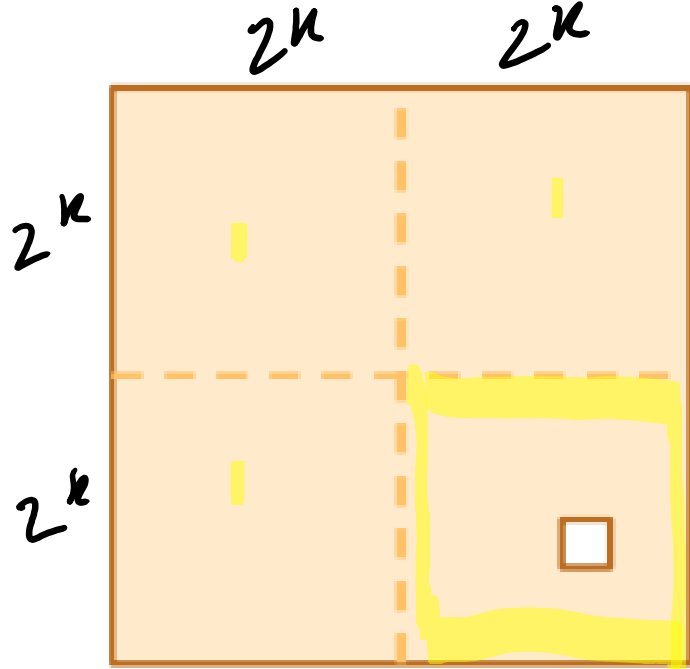
Checkerboard Tiling: Inductive Step

Now consider a $2^{k+1} \times 2^{k+1}$ board with one piece missing.



Checkerboard Tiling: Inductive Step

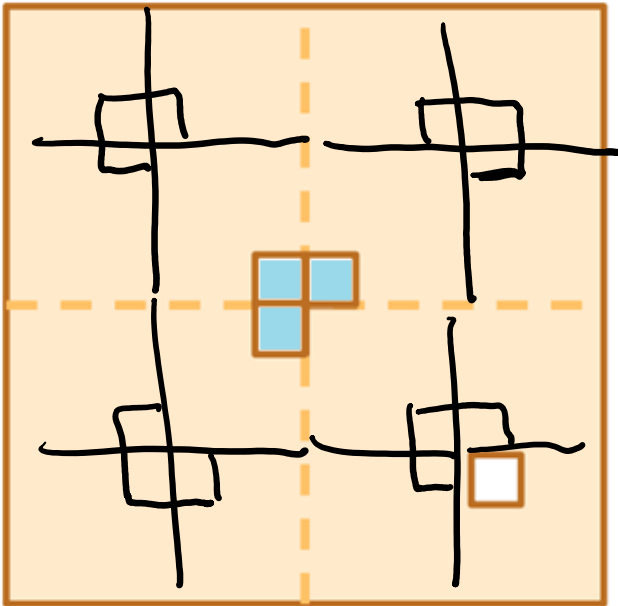
Now consider a $2^{k+1} \times 2^{k+1}$ board with one piece missing.



Divide the board into four quadrants of dimension $2^k \times 2^k$.

Checkerboard Tiling: Inductive Step

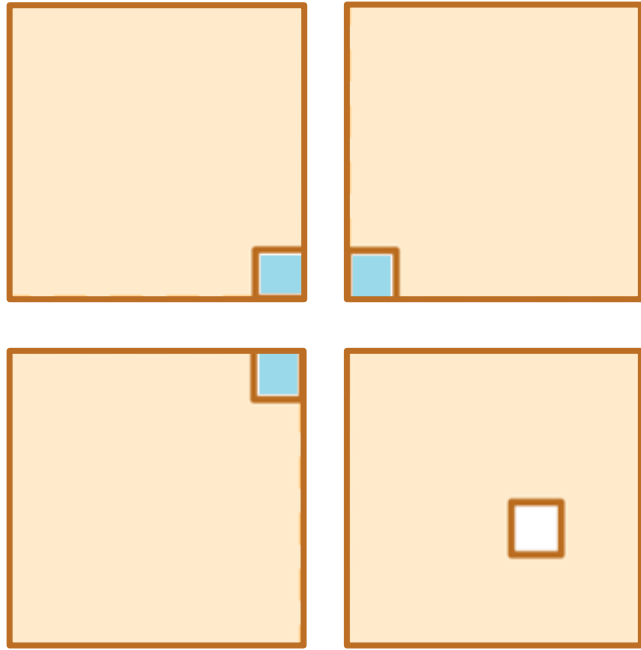
Now consider a $2^{k+1} \times 2^{k+1}$ board with one piece missing.



Place a single piece to occupy the three quadrants that aren't missing a piece.


Checkerboard Tiling: Inductive Step

Now consider a $2^{k+1} \times 2^{k+1}$ board with one piece missing.



Each quadrant is now a $2^k \times 2^k$ board with one piece missing.
We can tile each of these by the IH.

Checkerboard Tiling

1. Let $P(n)$ be "all $2^n \times 2^n$ boards with one square removed can be tiled with  pieces." We prove $P(n)$ for all integers $n \geq 1$ by induction.

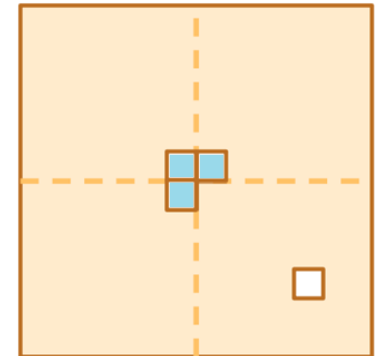
2. Base Case ($n = 1$): Observe that we can tile all 2×2 checkerboards:



So the base case holds.

3. Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary integer $k \geq 1$. That is, assume we can tile all $2^k \times 2^k$ checkerboards with one piece missing.

4. Inductive Step: We aim to show $P(k + 1)$. Consider an arbitrary $2^{k+1} \times 2^{k+1}$ checkerboard. We can divide the board into four quadrants, with one piece missing in one quadrant. Now place a single piece to occupy the three quadrants that aren't missing a piece. We now have four $2^k \times 2^k$ quadrants that are effectively each missing a piece. By the IH, we can tile each quadrant. Thus we can tile the entire checkerboard. So $P(k + 1)$ holds.



5. Conclusion: Thus $P(n)$ holds for all integers $n \geq 1$ by induction.