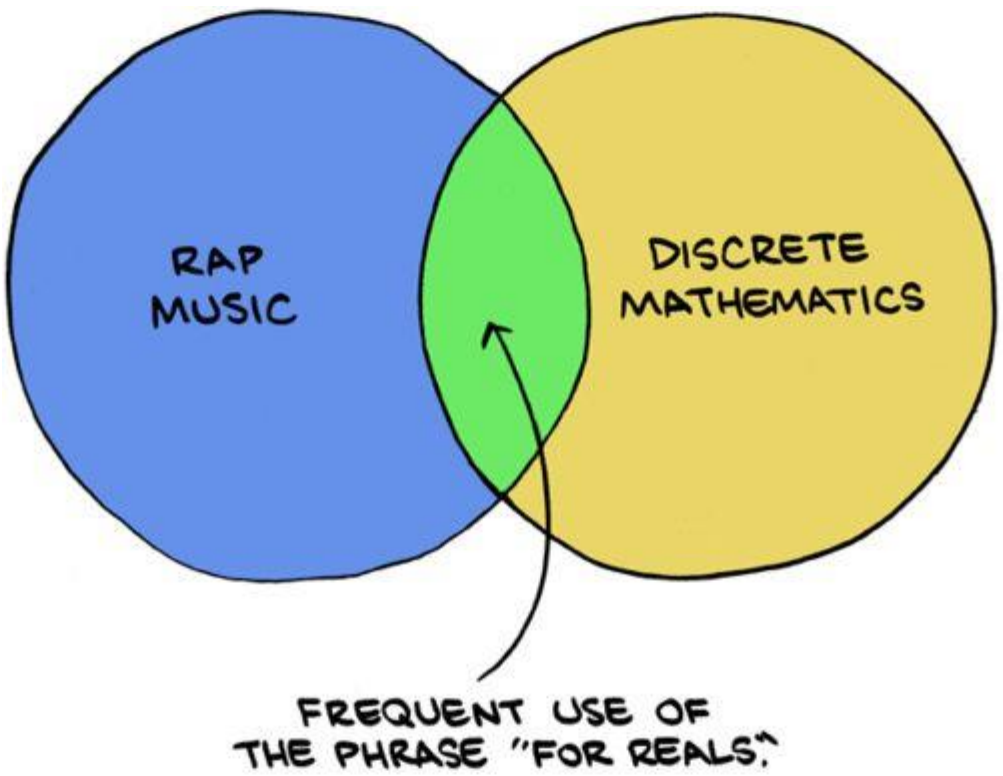


Set Theory Cont.

CSE 311: Foundations of Computing I
Lecture 12

Announcements

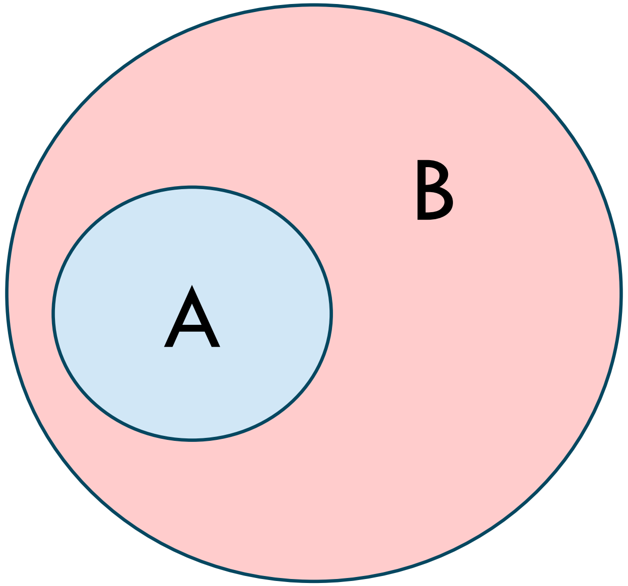
- Set Theory reference sheets are at the front for anyone who didn’t get them last week.
  All reference sheets are posted on the Resources tab of the website as well.
- HW3 solutions are at the front, grades to be posted tonight
- HW4 due 11:59 pm on Wednesday

Review

Sets

- A set is an unordered collection of distinct objects, called elements.
- The cardinality of a set is the number of elements in the set, denote $|S|$ for a set S.
- Sets A and B are equal if they have the same elements.
- Set A is a subset of B if every element of A is also in B.

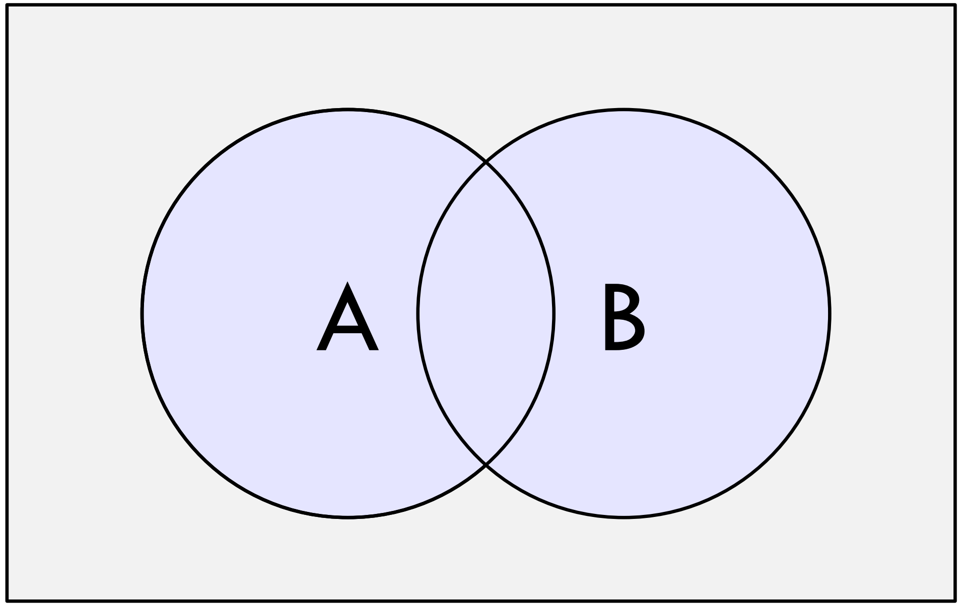
A is a subset of B



Set Operations

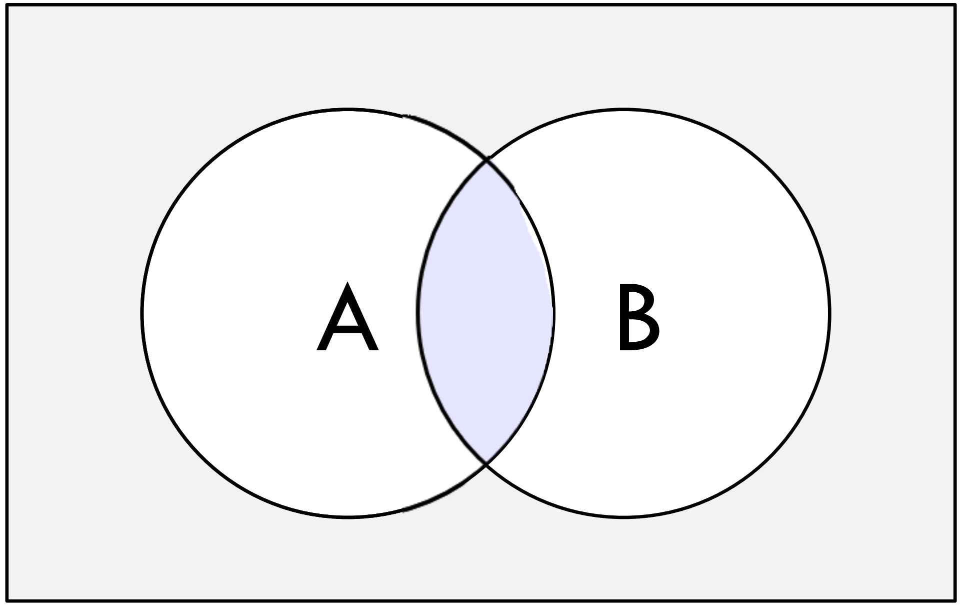
Union: $A \cup B$

$A \cup B = \{x : x \in A \vee x \in B\}$



Intersection: $A \cap B$

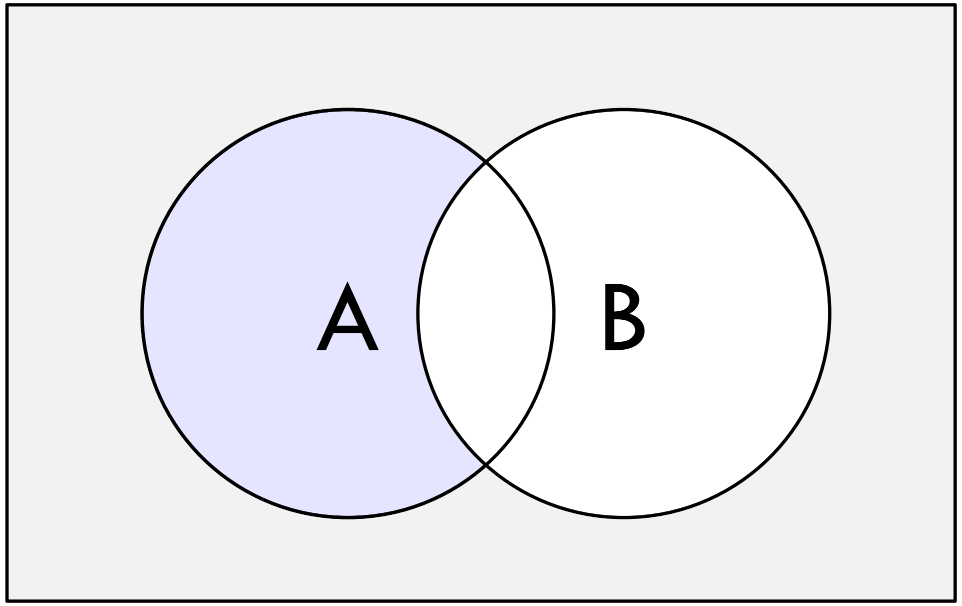
$A \cap B = \{x : x \in A \wedge x \in B\}$



Set Operations

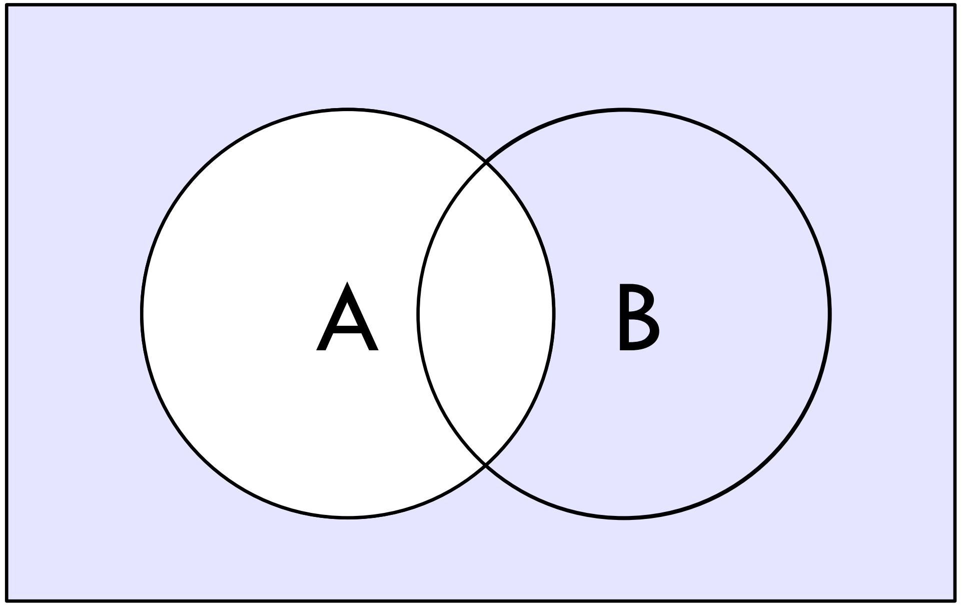
Set Difference: $A \setminus B$

$A \setminus B = \{x : x \in A \wedge x \notin B\}$



Set Complement: $\bar{A} = A^c$
(with respect to the universe $\mathcal{U}$)

$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$



Set Operations

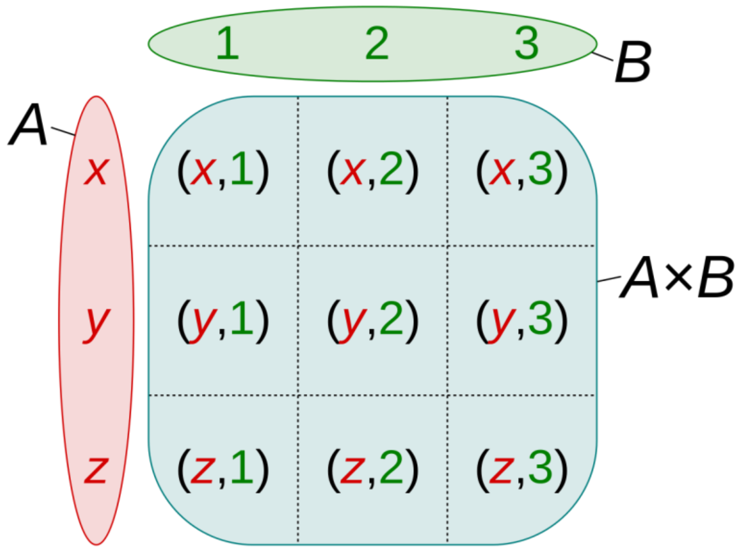
Powerset: $\mathcal{P}(A)$

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

$$\mathcal{P}(\{1,2,3\}) = \left\{ \begin{array}{c} \{\,\} \\ \{1\} \quad \{2\} \quad \{3\} \\ \{1,2\} \; \{1,3\} \; \{2,3\} \\ \{1,2,3\} \end{array} \right\}$$

Cartesian Product: $A \times B$

$$A \times B = \{(a, b) : a \in A, b \in B\}$$



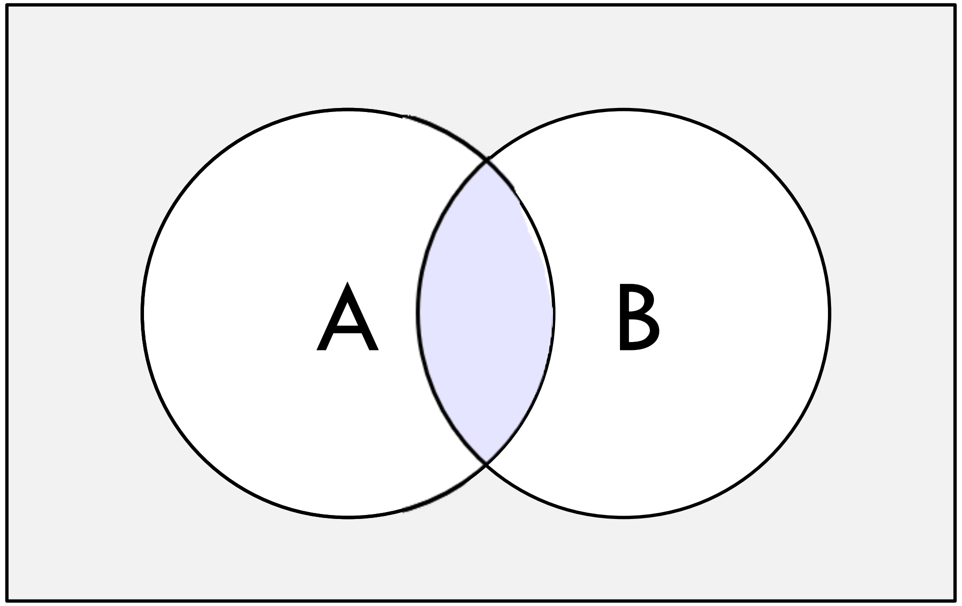
Subset Proofs

Claim 1

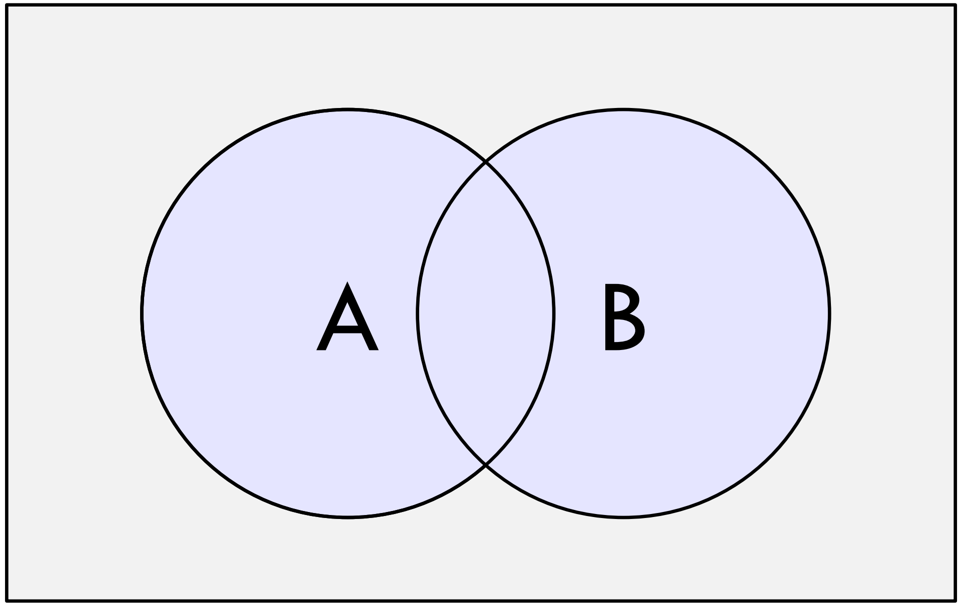
Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Intuition (Venn Diagram)



$\subseteq$



Claim 1

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Proof

Let A, B be arbitrary sets. Let $x \in A \cap B$ be arbitrary. Then by definition of intersection, $x \in A$ and $x \in B$. So certainly $x \in A$ or $x \in B$. Thus by definition of union, $x \in A \cup B$. Since x was arbitrary, $A \cap B \subseteq A \cup B$. Since A, B were arbitrary sets, the claim holds for all sets A, B.

Proving Subsets

To prove that $X \subseteq Y$, we let $x \in X$ be arbitrary and prove that $x \in Y$.

Claim 2

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

<u>Intuition (Example)</u>

$A = \{1,2\} \quad \mathcal{B} = \{1,2,3\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}\}$

Claim 2

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$\mathcal{P}(A) = \{B : B \subseteq A\}$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof

Let A, B be arbitrary sets. Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be arbitrary. Then by definition of powerset, $X \subseteq A$.

We know $X \subseteq A$ and $A \subseteq B$. We aim to show that $X \subseteq B$. Let $x \in X$ be arbitrary. Since $x \in X$ and $X \subseteq A$, then $x \in A$. Since $x \in A$ and $A \subseteq B$, then $x \in B$. Since x was arbitrary, $X \subseteq B$.

Now since we have that $X \subseteq B$, by definition of powerset $X \in \mathcal{P}(B)$. Since X was arbitrary, we have shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Symbols and Sets

Note that when writing set proofs, we follow various conventions.

We **DO** tend use symbols like $\in$, $\subseteq$, $\cup$, $\cap$, $\times$ etc. (instead of writing out the symbol in English).
E.g. "Let $x \in A$ be arbitrary"

We **DO NOT** tend to use symbols like $\wedge$, $\vee$, $\neg$ (but rather write them out in English).**
E.g. "Then $x \in A$ and $x \in B$"

**There are exceptions to this if logical symbols provide clarity when applying equivalence rules (Absorption, DeMorgan's Laws, etc.). The proof of Claim 3 will be an example of that.

Exercises

Which of the following statements are true?

If $x \in A \cap B$ then $(x \in A) \cap (x \in B)$.

False. This should be $(x \in A) \wedge (x \in B)$.

If $x \in C \backslash D$ then $x \in C \wedge \neg(x \in D)$.

True.

If $X \subseteq \mathcal{P}(A)$ then $X \in A$.

False. This should be if $X \in \mathcal{P}(A)$ then $X \subseteq A$.

If $(a, b) \in E \times F$ then $a \in E$ and $b \in F$.

True.

Set Equality Proofs

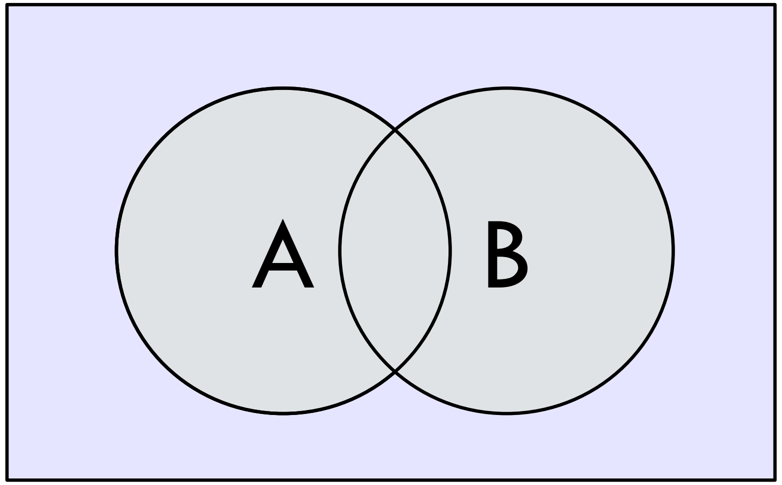
Claim 3 (DeMorgan's Law for Sets)

Definitions
$A = B \equiv A \subseteq B \land B \subseteq A$
$A \cup B = \{x : x \in A \lor x \in B\}$
$A \cap B = \{x : x \in A \land x \in B\}$
$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

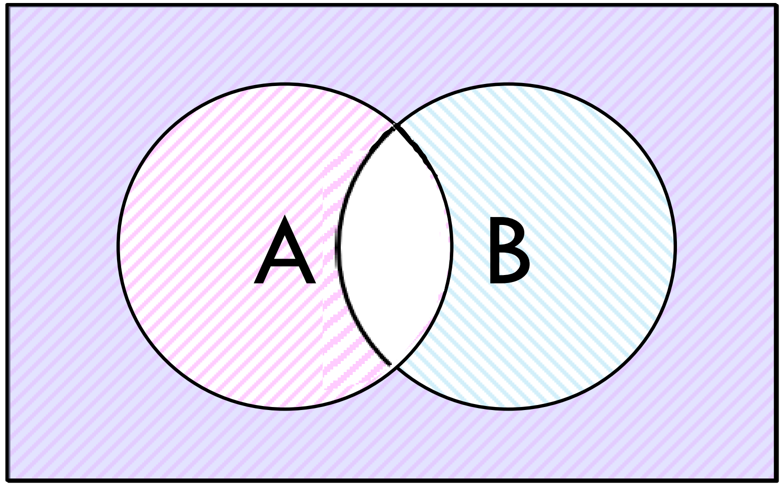
Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

<u>Intuition (Venn Diagram)</u>

$\overline{A \cup B}$



$\bar{A} \cap \bar{B}$



Claim 3

Definitions
$A = B \equiv A \subseteq B \wedge B \subseteq A$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$
$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

<u>Proof Strategy</u>

- Let A, B be arbitrary sets.
- Prove that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.
- Prove that $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Definitions
$A = B \equiv A \subseteq B \wedge B \subseteq A$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$
$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$

<u>Proof (Method 1)</u>

Let A, B be arbitrary sets.

$\Rightarrow$ First we show that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$. Let $x \in \overline{A \cup B}$ be arbitrary. By definition of complement, we have that $\neg(x \in A \cup B)$. Then by definition of union, $\neg(x \in A \vee x \in B)$. So by DeMorgan's Law, $x \notin A \wedge x \notin B$. Then by definition of complement, $x \in \bar{A}$ and $x \in \bar{B}$. By definition of intersection, $x \in \bar{A} \cap \bar{B}$. Since x was arbitrary, $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

$\Leftarrow$ Now we show that $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$. Let $x \in \bar{A} \cap \bar{B}$ be arbitrary. By definition of intersection, we have that $x \in \bar{A}$ and $x \in \bar{B}$. By definition of complement, we have $\neg(x \in A) \wedge \neg(x \in B)$. Apply DeMorgan's Law, we have $\neg(x \in A \vee x \in B)$. Then by definition of union, $\neg(x \in A \cup B)$. Then by definition of complement, $x \in \overline{A \cup B}$. Since x was arbitrary, $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Thus we have shown that $\overline{A \cup B} = \bar{A} \cap \bar{B}$. Since A, B were arbitrary, the claim holds.

Proving Set Equality: Method 1

One way to prove that $X = Y$ is to show that $X \subseteq Y$ by one subset proof, and $Y \subseteq X$ by another subset proof.

Proving Set Equality

- In the previous example, there was a lot of repetitive work to show $\Rightarrow$ and $\Leftarrow$
- Can we show $\Leftrightarrow$ directly?

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Proof (Using Method 2)

Let A, B be arbitrary sets. Let x be arbitrary. Then:

$x \in \overline{A \cup B} \equiv \neg(x \in A \cup B)$	Definition of Complement
$\equiv \neg(x \in A \vee x \in B)$	Definition of Union
$\equiv \neg(x \in A) \wedge \neg(x \in B)$	DeMorgan's Law
$\equiv x \in \bar{A} \wedge x \in \bar{B}$	Definition of Complement
$\equiv x \in \overline{A \cap B}$	Definition of Intersection

Since x was arbitrary, we have shown that $\overline{A \cup B} = \bar{A} \cap \bar{B}$. Since A, B were arbitrary sets, the claim holds.

Proving Set Equality: Method 2

Another way to prove that $X = Y$ is to show that for arbitrary element x, $x \in X \equiv \cdots \equiv x \in Y$ by a chain of equivalences.

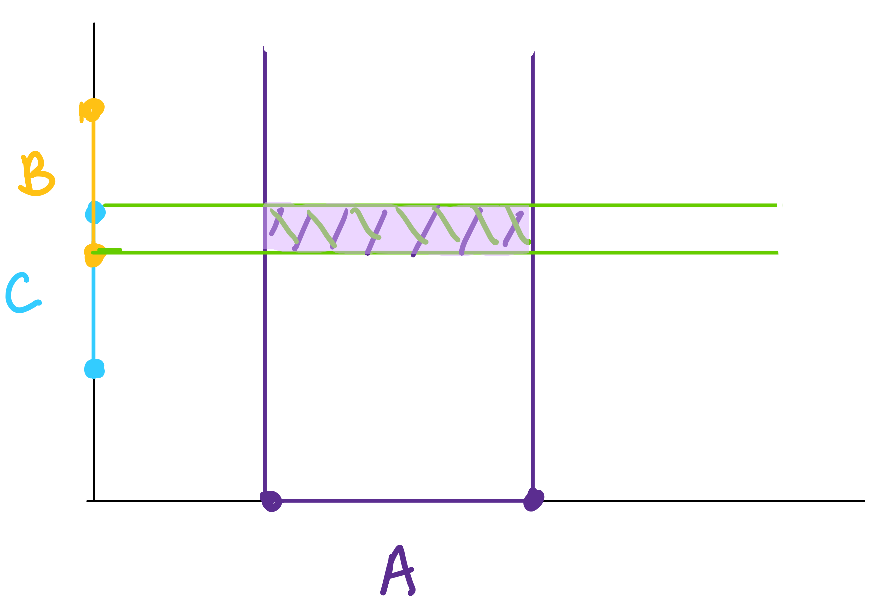
Claim 4

Definitions
$A \times B = \{(a, b): a \in A, b \in B\}$
$A \cup B = \{x : x \in A \lor x \in B\}$

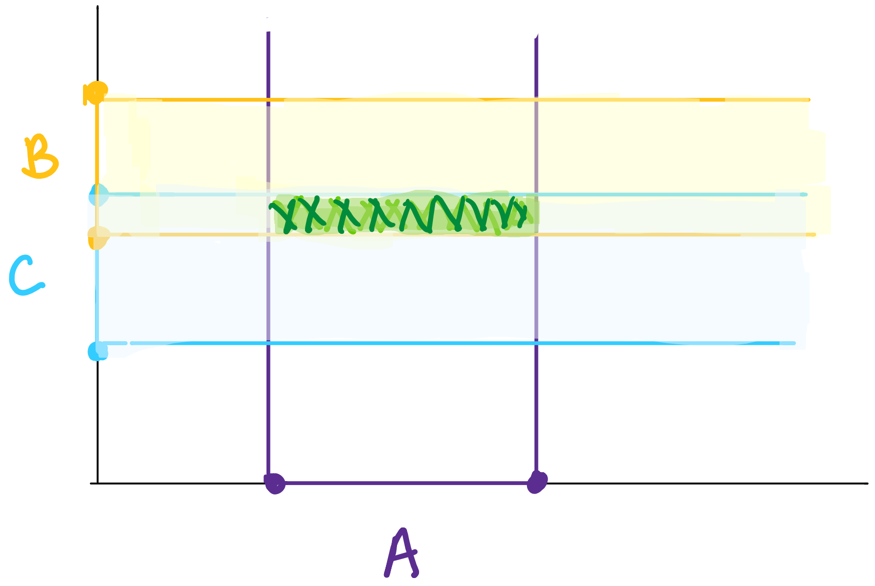
Claim 4: For all sets A, B, C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

<u>Intuition (Diagram)</u>

$A \times (B \cap C)$



$(A \times B) \cap (A \times C)$



Claim 4

Definitions
$A \times B = \{(a, b): a \in A, b \in B\}$
$A \cup B = \{x : x \in A \vee x \in B\}$

Claim 4: For all sets A, B, C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof (Method 2)

Let A, B, C be arbitrary sets. Let (y, z) be an arbitrary pair. Then:

$(y, z) \in A \times (B \cap C) \equiv (y \in A) \wedge (z \in B \cap C)$	Def. of ×
$\equiv (y \in A) \wedge \big((z \in B) \wedge (z \in C)\big)$	Def. of ∩
$\equiv (y \in A) \wedge (y \in A) \wedge (z \in B) \wedge (z \in C)$	Idempotency
$\equiv (y \in A) \wedge (z \in B) \wedge (y \in A) \wedge (z \in C)$	Assoc. and Commutativity
$\equiv (y, z) \in A \times B \wedge (y, z) \in A \times C$	Def of ×
$\equiv (y, z) \in (A \times B) \cap (A \times C)$	Def of ∩

Since (y, z) was arbitrary, we have shown that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. Since A, B, C were arbitrary sets, the claim holds.

Exercises

Which of the following statements are true?

For all sets A, B: $A \cup (A \cap B) = A$.

True.

For all sets A, B, C if $A \subseteq B \cup C$ then $A \subseteq B$ or $A \subseteq C$.

False. Consider $A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$.

For all sets A, B, C if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

True.

For all sets A, B, C: $(A \cup B) \times C = (A \times C) \cup (B \times C)$

True.