

Set Theory Cont.

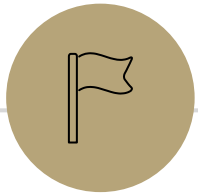
CSE 311: Foundations of
Computing I
Lecture 12

Announcements

- Set Theory reference sheets are at the front for anyone who didn't get them last week.

All reference sheets are posted on the Resources tab of the website as well.

- HW3 solutions are at the front, grades to be posted tonight
- HW4 due 11:59 pm on Wednesday

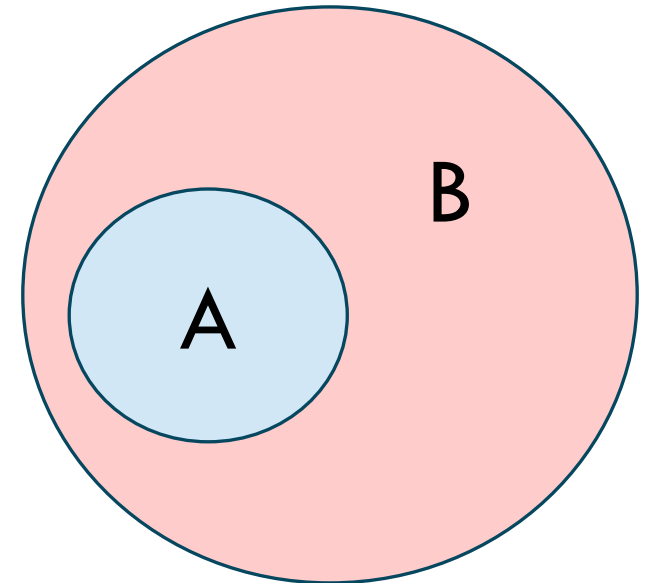


Review

Sets

- A **set** is an unordered collection of distinct objects, called elements.
- The **cardinality** of a set is the number of elements in the set, denote $|S|$ for a set S .
- Sets A and B are **equal** if they have the same elements.
- Set A is a **subset** of B if every element of A is also in B .

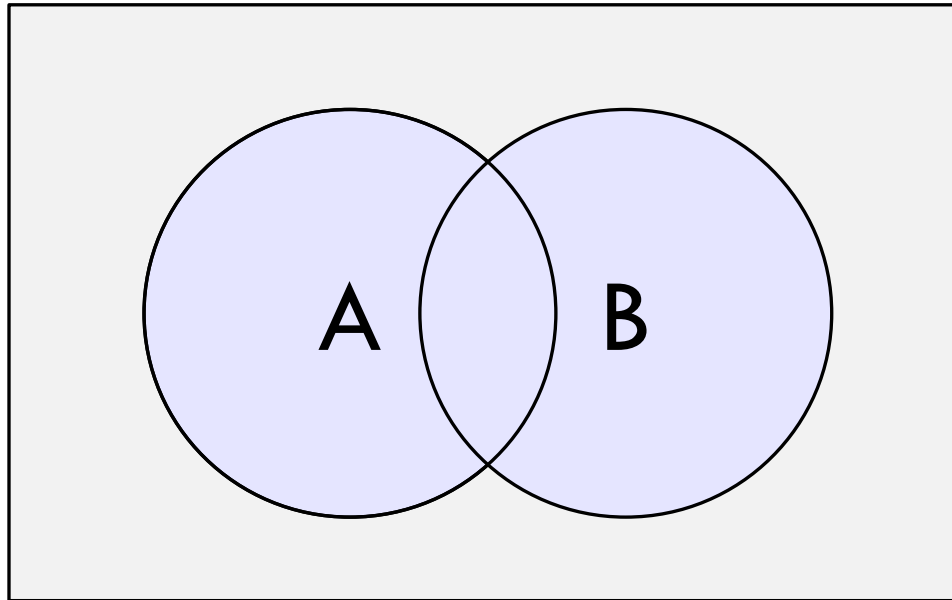
A is a subset of B



Set Operations

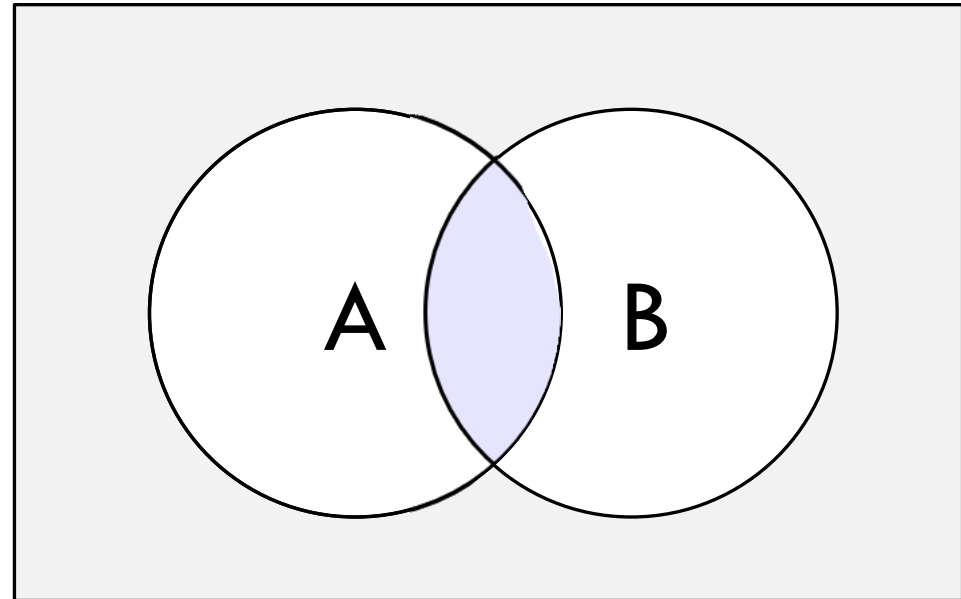
Union: $A \cup B$

$$A \cup B = \{x : x \in A \vee x \in B\}$$



Intersection: $A \cap B$

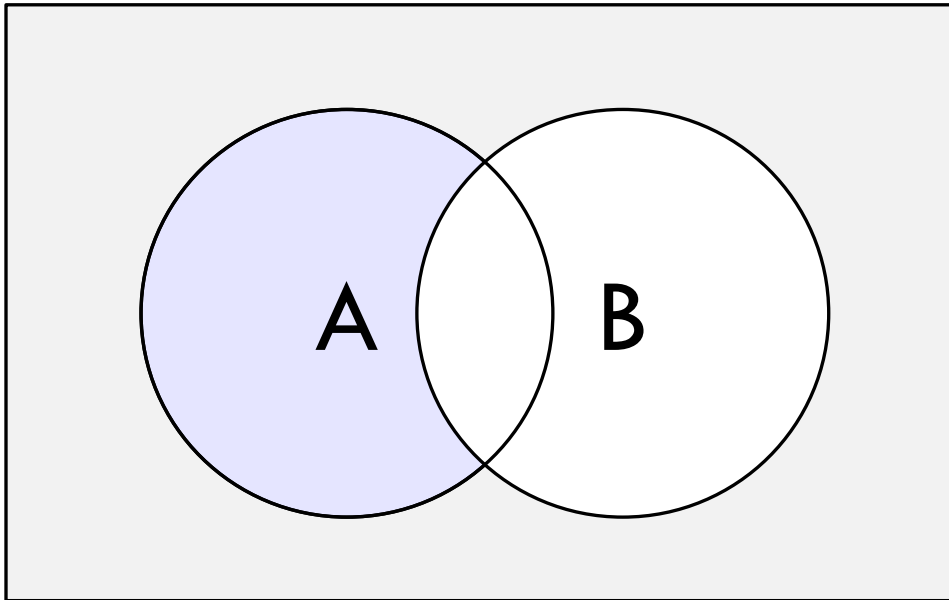
$$A \cap B = \{x : x \in A \wedge x \in B\}$$



Set Operations

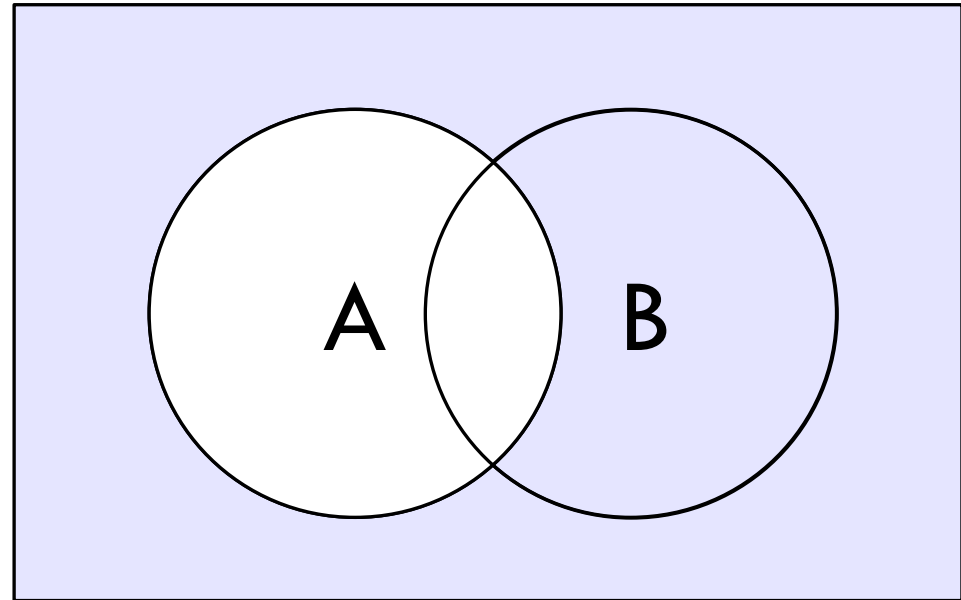
Set Difference: $A \setminus B$

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$



Set Complement: $\bar{A} = A^c$
(with respect to the universe \mathcal{U})

$$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$$



Set Operations

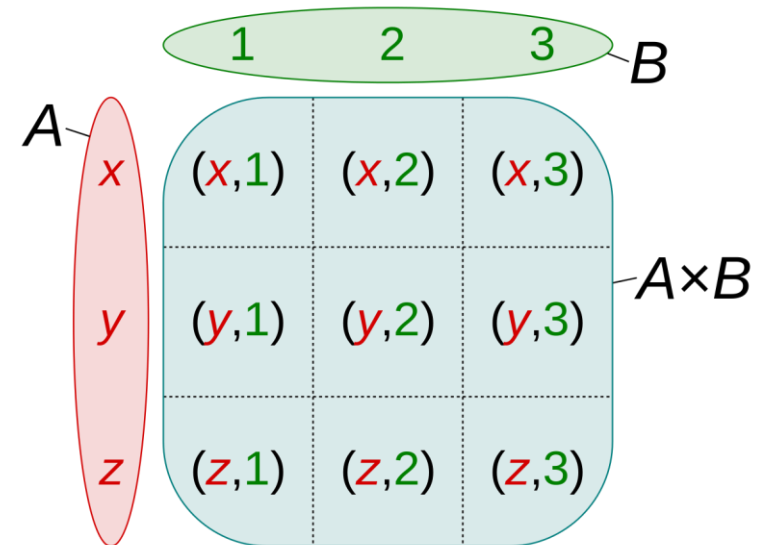
Powerset: $\mathcal{P}(A)$

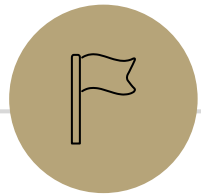
$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

$$\mathcal{P}(\{1,2,3\}) = \left\{ \begin{array}{l} \{\} \\ \{1\} \quad \{2\} \quad \{3\} \\ \{1,2\} \quad \{1,3\} \quad \{2,3\} \\ \{1,2,3\} \end{array} \right\}$$

Cartesian Product: $A \times B$

$$A \times B = \{(a,b) : a \in A, b \in B\}$$





Subset Proofs

Claim 1

Claim 1: For all sets A, B , we have $A \cap B \subseteq A \cup B$.

Intuition (Venn Diagram)

Definitions

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

Claim 1

Claim 1: For all sets A, B , we have $A \cap B \subseteq A \cup B$.

Proof

Definitions

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

Proving Subsets

To prove that $X \subseteq Y$, we

Claim 2

Definitions

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Intuition (Example)

Claim 2

Definitions

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof

Symbols and Sets

Note that when writing set proofs, we follow various conventions.

We **DO** tend use symbols like \in , \subseteq , \cup , \cap , \times etc. (instead of writing out the symbol in English).

E.g. "Let $x \in A$ be arbitrary"

We **DO NOT** tend to use symbols like \wedge , \vee , \neg (but rather write them out in English).**

E.g. "Then $x \in A$ and $x \in B$ "

**There are exceptions to this if logical symbols provide clarity when applying equivalence rules (Absorption, DeMorgan's Laws, etc.). The proof of Claim 3 will be an example of that.

Exercises

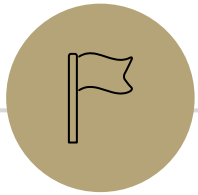
Which of the following statements are true?

If $x \in A \cap B$ then $(x \in A) \cap (x \in B)$.

If $x \in C \setminus D$ then $(x \in C) \wedge \neg(x \in D)$.

If $X \subseteq \mathcal{P}(A)$ then $X \in A$.

If $(a, b) \in E \times F$ then $a \in E$ and $b \in F$.



Set Equality Proofs

Claim 3 (DeMorgan's Law for Sets)

Claim 3: For all sets A, B , $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Intuition (Venn Diagram)

Definitions

$$A = B \equiv A \subseteq B \wedge B \subseteq A$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$$

Claim 3

Claim 3: For all sets A, B , $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof Strategy

- Let A, B be arbitrary sets.
- Prove that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.
- Prove that $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Definitions

$$A = B \equiv A \subseteq B \wedge B \subseteq A$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$$

Claim 3

Claim 3: For all sets A, B , $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Proof (Method 1)

Definitions

$$A = B \equiv A \subseteq B \wedge B \subseteq A$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$$

Proving Set Equality: Method 1

One way to prove that $X = Y$ is to show that _____

Proving Set Equality

- In the previous example, there was a lot of repetitive work to show \Rightarrow and \Leftarrow
- Can we show \Leftrightarrow directly?

Claim 3

Claim 3: For all sets A, B , $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Proof (Using Method 2)

Proving Set Equality: Method 2

Another way to prove that $X = Y$ is to show that _____

Claim 4

Definitions

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

Claim 4: For all sets A, B, C , $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Intuition (Diagram)

Claim 4

Definitions

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

Claim 4: For all sets A, B, C , $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof (Method 2)

Exercises

Which of the following statements are true?

For all sets A, B : $A \cup (A \cap B) = A$.

For all sets A, B, C if $A \subseteq B \cup C$ then $A \subseteq B$ or $A \subseteq C$.

For all sets A, B, C if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

For all sets A, B, C : $(A \cup B) \times C = (A \times C) \cup (B \times C)$