

# Set Theory Cont.

CSE 311: Foundations of  
Computing I  
Lecture 12

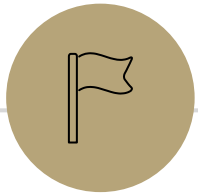
# Announcements

- Set Theory reference sheets are at the front for anyone who didn't get them last week.

All reference sheets are posted on the Resources tab of the website as well.

- HW3 solutions are at the front, grades to be posted tonight
- HW4 due 11:59 pm on Wednesday

**Hint**



# Review

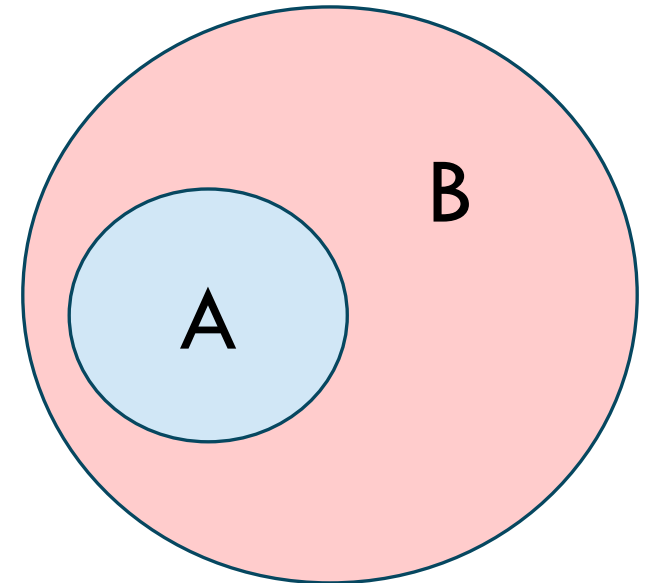


# Sets

$$A \neq B \quad |S| \quad \equiv_m \quad x \in A \quad x \notin A$$
$$\equiv$$

- A **set** is an unordered collection of distinct objects, called elements.
- The **cardinality** of a set is the number of elements in the set, denote  $|S|$  for a set  $S$ .
- Sets  $A$  and  $B$  are **equal** if they have the same elements.  $A = B$
- Set  $A$  is a **subset** of  $B$  if every element of  $A$  is also in  $B$ .  $A \subseteq B$

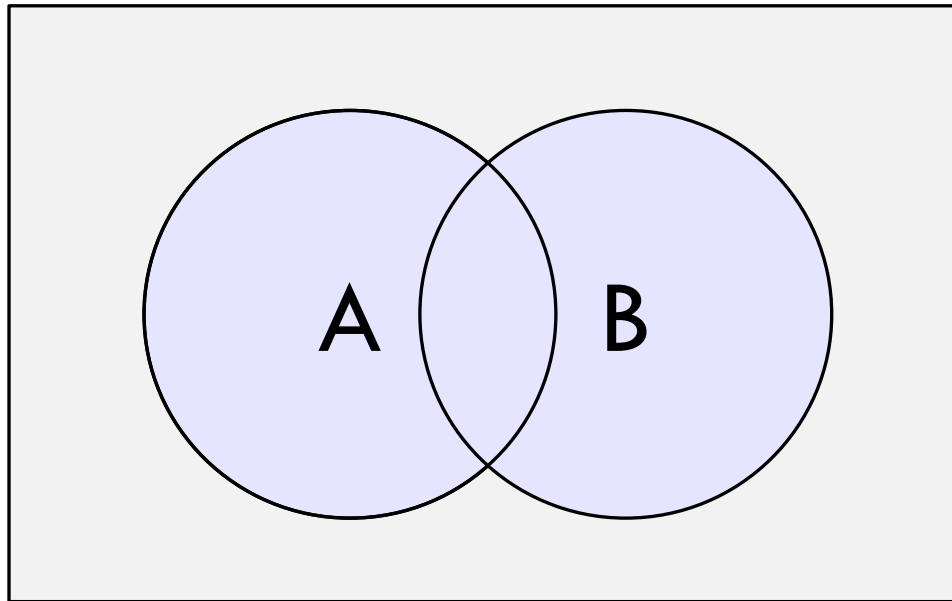
A is a subset of B



# Set Operations

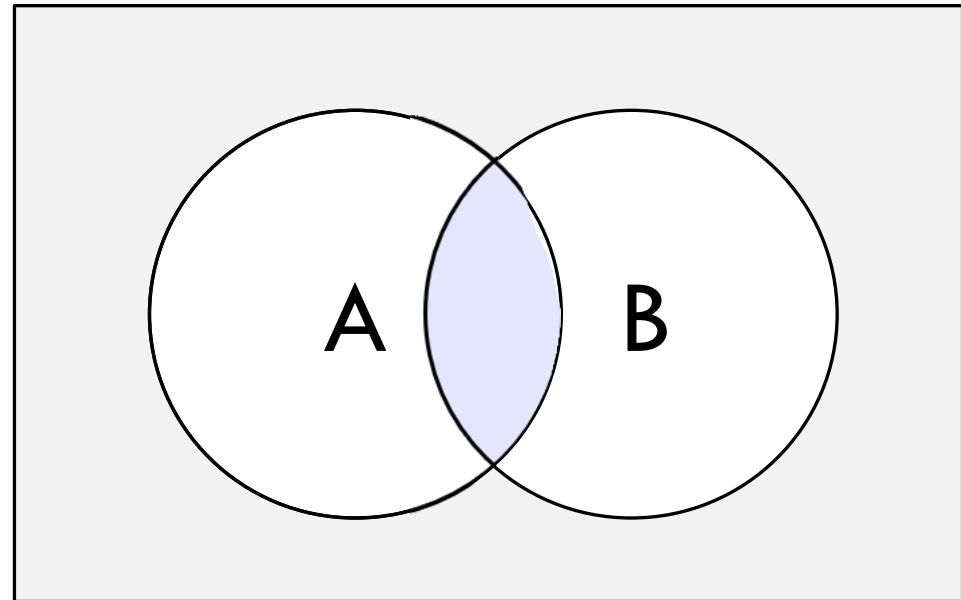
Union:  $A \cup B$

$$A \cup B = \{x : x \in A \vee x \in B\}$$



Intersection:  $A \cap B$

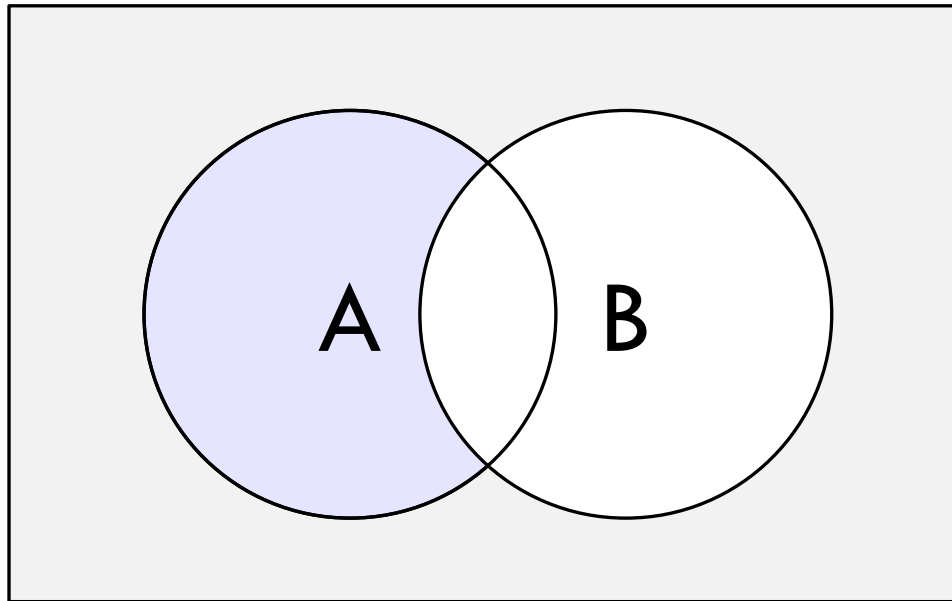
$$A \cap B = \{x : x \in A \wedge x \in B\}$$



# Set Operations

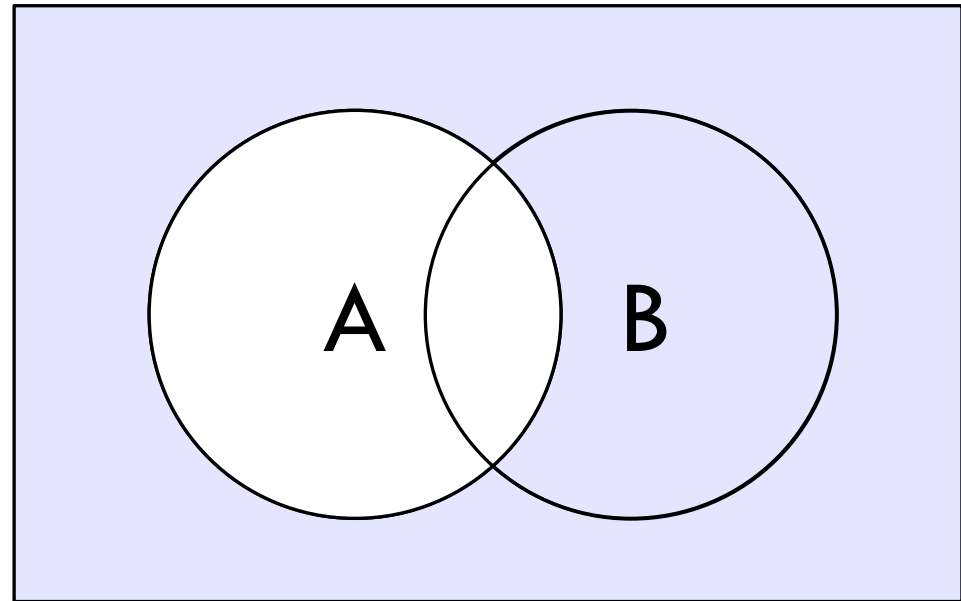
Set Difference:  $A \setminus B$

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$



Set Complement:  $\bar{A} = A^c$   
(with respect to the universe  $\mathcal{U}$ )

$$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$$



# Set Operations

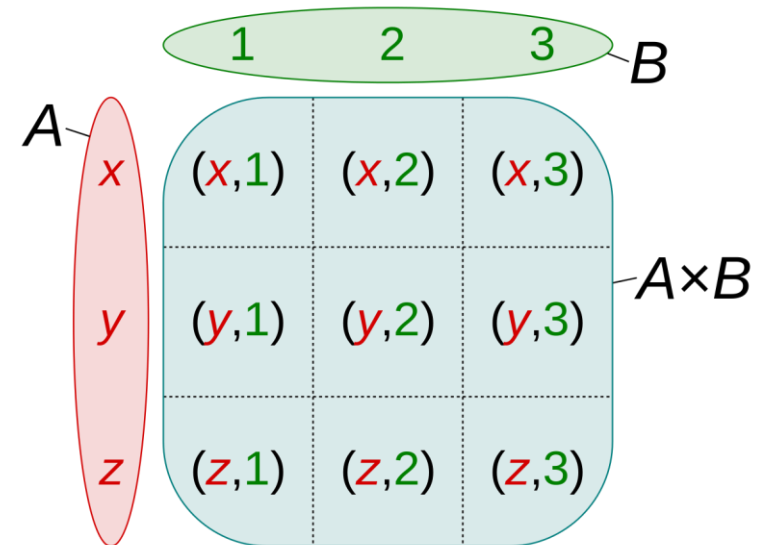
Powerset:  $\mathcal{P}(A)$

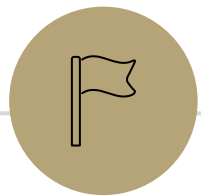
$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

$$\mathcal{P}(\{1,2,3\}) = \left\{ \begin{array}{l} \{\} \\ \{1\} \quad \{2\} \quad \{3\} \\ \{1,2\} \quad \{1,3\} \quad \{2,3\} \\ \{1,2,3\} \end{array} \right\}$$

Cartesian Product:  $A \times B$

$$A \times B = \{(a,b) : a \in A, b \in B\}$$





## Subset Proofs

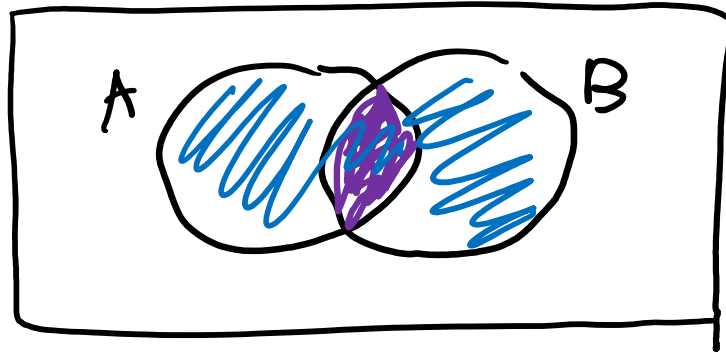
$$X \subseteq Y$$

$$X = Y$$

# Claim 1

Claim 1: For all sets  $A, B$ , we have  $A \cap B \subseteq A \cup B$ .

Intuition (Venn Diagram)



## Definitions

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

# Claim 1

## Definitions

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$$A \cup B = \overline{\{x : x \in A \vee x \in B\}}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

Claim 1: For all sets  $A, B$ , we have  $A \cap B \subseteq A \cup B$ .

## Proof

Let  $A, B$  be arbitrary sets. Let  $x$  be arbitrary. Suppose  $x \in A \cap B$ . By def of intersection,  $x \in A$  and  $x \in B$ . Clearly,  $x \in A$  or  $x \in B$ . By def of union,  $x \in A \cup B$ . Since  $x$  was arbitrary,  $A \cap B \subseteq A \cup B$ . Since  $A, B$  were arbitrary, the

claim holds for all sets  $A, B$ .

# Proving Subsets

To prove that  $X \subseteq Y$ , we let  $x \in X$  be arbitrary, and  
prove  $x \in Y$ .

# Claim 2

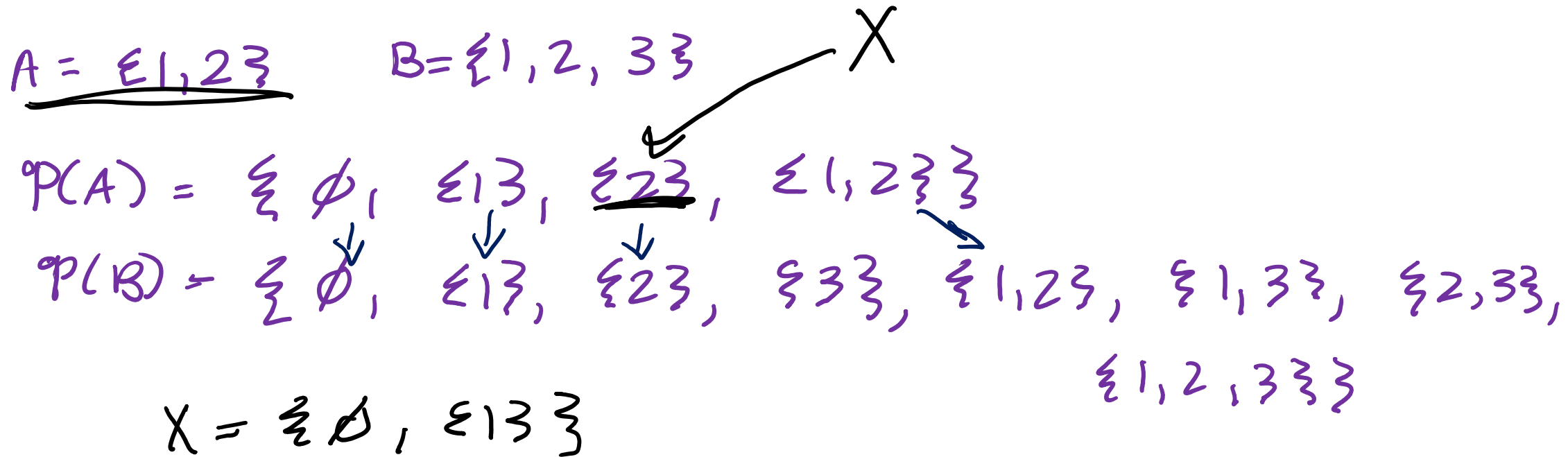
## Definitions

$$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$$

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Claim 2: For all sets  $A, B$  if  $A \subseteq B$  then  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ .

Intuition (Example)



## Claim 2

$$\forall X (X \in \mathcal{P}(A) \rightarrow X \in \mathcal{P}(B))$$

### Definitions

$$A \subseteq B \equiv \forall x (x \in A \rightarrow x \in B)$$

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Claim 2: For all sets  $A, B$  if  $A \subseteq B$  then  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ .

### Proof

Let  $A, B$  be arbitrary sets. Suppose  $A \subseteq B$ . Let  $X$  be arbitrary. Suppose  $X \in \mathcal{P}(A)$ . Then by definition of powerset,  $X \subseteq A$ .

We have  $X \subseteq A$  and  $A \subseteq B$ . We aim to show  $X \subseteq B$ . Let  $y \in X$  be arbitrary. Since  $y \in X$  and  $X \subseteq A$ , then  $y \in A$ . Since  $y \in A$  and  $A \subseteq B$  then  $y \in B$ . Since  $y$  is arbitrary,  $X \subseteq B$ .

By def of powerset, since  $X \subseteq B$ ,  $X \in \mathcal{P}(B)$ . Since  $X$  was arbitrary,  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ . Since  $A, B$  were arbitrary, the claim holds.

# Symbols and Sets

Note that when writing set proofs, we follow various conventions.

We **DO** tend use symbols like  $\in$ ,  $\subseteq$ ,  $\cup$ ,  $\cap$ ,  $\times$  etc. (instead of writing out the symbol in English).

E.g. "Let  $x \in A$  be arbitrary"

We **DO NOT** tend to use symbols like  $\wedge$ ,  $\vee$ ,  $\neg$  (but rather write them out in English).\*\*

E.g. "Then  $x \in A$  and  $x \in B$ "

$$\neg (x \in A \wedge x \in B)$$

\*\*There are exceptions to this if logical symbols provide clarity when applying equivalence rules (Absorption, DeMorgan's Laws, etc.). The proof of Claim 3 will be an example of that.

# Exercises

Which of the following statements are true?

If  $x \in \underline{A \cap B}$  then  $\underline{(x \in A)} \cap \underline{(x \in B)}$ .      False  
 $(x \in A) \wedge (x \in B)$

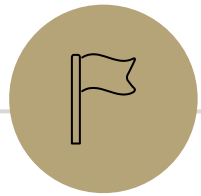
If  $x \in C \setminus D$  then  $(x \in C) \wedge \underline{\neg(x \in D)}$ .      True  
 $x \in C$  and  $x \notin D$ .

If  $X \subseteq \mathcal{P}(A)$  then  $\underline{X \in A}$ .      False

If  $X \in \mathcal{P}(A)$ , then  $X \subseteq A$ .  
True

If  $\underline{(a, b) \in E \times F}$  then  $\underline{a \in E}$  and  $\underline{b \in F}$ .

True.



# Set Equality Proofs



# Claim 3 (DeMorgan's Law for Sets)

Claim 3: For all sets  $A, B$ ,  $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Intuition (Venn Diagram)



## Definitions

$$A = B \equiv A \subseteq B \wedge B \subseteq A$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$\bar{A} = \{x \in U : x \notin A\}$$

# Claim 3

Claim 3: For all sets  $A, B$ ,  $\overline{A \cup B} = \bar{A} \cap \bar{B}$

## Proof Strategy

- Let  $A, B$  be arbitrary sets.
- Prove that  $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$ .
- Prove that  $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$ .

### Definitions

$$A = B \equiv A \subseteq B \wedge B \subseteq A$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$$

# Claim 3

Claim 3: For all sets  $A, B$ ,  $\overline{A \cup B} = \bar{A} \cap \bar{B}$ .

Proof (Method 1)

Let  $A, B$  be arbitrary sets.

$\subseteq$  First we show  $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$ . Let  $x \in \overline{A \cup B}$  be arbitrary. By def of complement  $\neg(x \in A \cup B)$ . By def of union,  $\neg(x \in A \vee x \in B)$ . By De Morgan's Law,  $x \notin A \wedge x \notin B$ . By def of complement,  $x \in \bar{A}$  and  $x \in \bar{B}$ . By def of intersection,  $x \in \bar{A} \cap \bar{B}$ . Since  $x$  was arbitrary,  $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$ .

$\supseteq$  Next we show  $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$ . Let  $x \in \bar{A} \cap \bar{B}$  be arbitrary. By def of intersection,  $x \in \bar{A}$  and  $x \in \bar{B}$ . So by def of comp.

$\neg(x \in A) \wedge \neg(x \in B)$ . By De Morgan's Law,  $\neg(x \in A \vee x \in B)$ . By def of union,  $\neg(x \in A \cup B)$ . By def of complement,  $x \in \overline{A \cup B}$ .

## Definitions

$$A = B \equiv A \subseteq B \wedge B \subseteq A$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$$

# Proving Set Equality: Method 1

One way to prove that  $X = Y$  is to show that  $X \subseteq Y$  and  
 $Y \subseteq X$ .

# Proving Set Equality

- In the previous example, there was a lot of repetitive work to show  $\Rightarrow$  and  $\Leftarrow$
- Can we show  $\Leftrightarrow$  directly?

# Claim 3

Claim 3: For all sets  $A, B$ ,  $\overline{A \cup B} = \bar{A} \cap \bar{B}$ .

Proof (Using Method 2)

Let  $A, B$  be arbitrary sets. Let  $x$  be an arbitrary object. Then:

$$\underline{x \in \overline{A \cup B}} \equiv \underline{\neg (x \in A \cup B)}$$

Def of Complement

$$\equiv \underline{\neg (x \in A \vee x \in B)}$$

Def of Union

$$\equiv \neg (x \in A) \wedge \neg (x \in B)$$

DeMorgan's Law

$$\equiv x \in \bar{A} \wedge x \in \bar{B}$$

Def of Complement

$$\equiv x \in \bar{A} \cap \bar{B}$$

Def of Intersection

Since  $x$  was arbitrary, we have shown  $\overline{A \cup B} = \bar{A} \cap \bar{B}$ .

Since  $A, B$  were arbitrary, the claim holds.

## Proving Set Equality: Method 2

Another way to prove that  $X = Y$  is to show that for an  
arbitrary element  $x$ ,  $x \in X \equiv \dots \equiv x \in Y$   
by a chain of equivalences.

# Claim 4

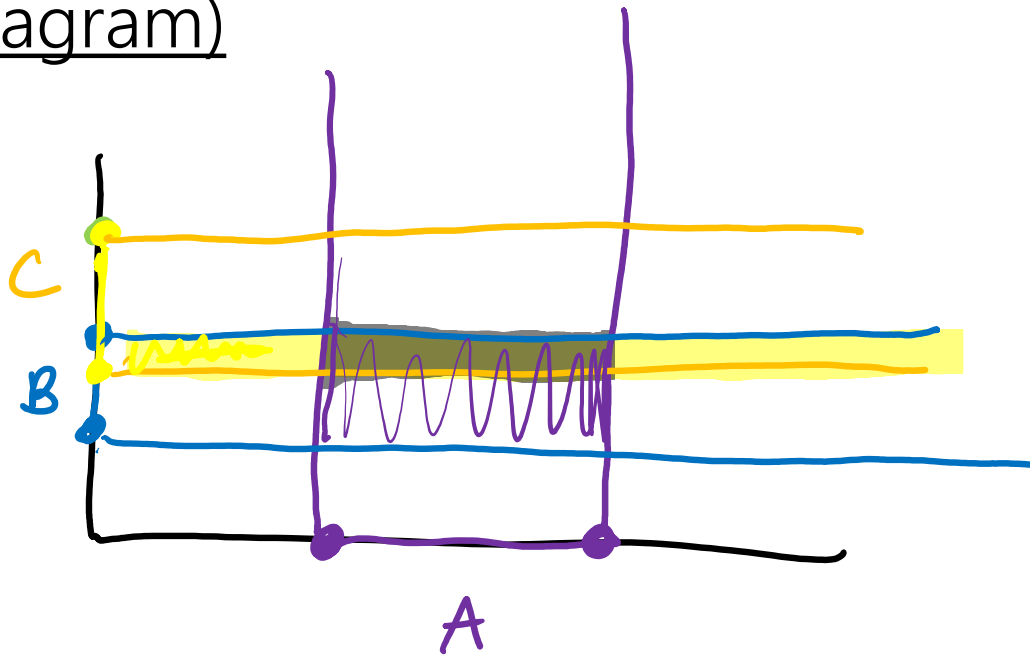
## Definitions

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

Claim 4: For all sets  $A, B, C$ ,  $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Intuition (Diagram)



# Claim 4

## Definitions

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

Claim 4: For all sets  $A, B, C$ ,  $A \times (B \cap C) = (A \times B) \cap (A \times C)$ .

Proof (Method 2)

Let  $A, B, C$  be arbitrary sets. Let  $(y, z)$  be an arbitrary pair.

$$\begin{aligned} (y, z) \in A \times (B \cap C) &\equiv y \in A \wedge z \in B \cap C && \text{Def of } \times \\ &\equiv y \in A \wedge z \in B \wedge z \in C && \text{Def of } \cap \\ &\equiv y \in A \wedge y \in A \wedge z \in B \wedge z \in C && \text{Idempotency} \\ &\equiv y \in A \wedge z \in B \wedge y \in A \wedge z \in C && \text{ASSOC \& Commut} \\ &\equiv (y, z) \in (A \times B) \wedge (y, z) \in A \times C && \text{Def of } \times \\ &\equiv (y, z) \in (A \times B) \cap (A \times C) && \text{Def of } \cap \end{aligned}$$

# Exercises

Which of the following statements are true?

For all sets  $A, B$ :  $A \cup (A \cap B) = A$ .

For all sets  $A, B, C$  if  $A \subseteq B \cup C$  then  $A \subseteq B$  or  $A \subseteq C$ .

For all sets  $A, B, C$  if  $A \subseteq B$  and  $A \subseteq C$  then  $A \subseteq B \cap C$ .

For all sets  $A, B, C$ :  $(A \cup B) \times C = (A \times C) \cup (B \times C)$