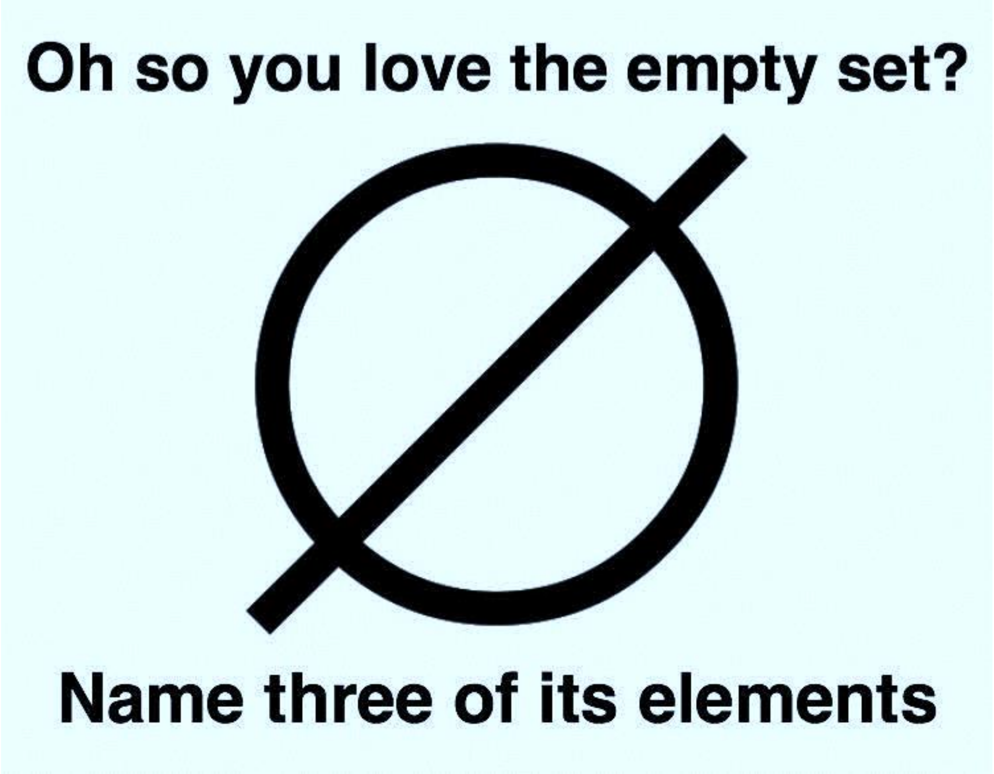


Set Theory

CSE 311: Foundations of Computing I
Lecture 11

Announcements

- HW3 late deadline is today at 11:59 pm
- HW4 has been posted, due 11:59 pm on Wednesday

Wrapping Number Theory

Recall

Let a, b, c, d and $m > 0$ be integers.

- If $a \equiv_m b$, then $b \equiv_m a$.
- If $a \equiv_m b$ and $c \equiv_m d$, then $a + c \equiv_m b + d$.
- If $a \equiv_m b$ and $c \equiv_m d$, then $ac \equiv_m bd$.
- If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.
- $a \equiv_m b$ if and only if $a \,\%\, m = b \,\%\, m$.

Solving in Modular Arithmetic

Solve: $5 + x \equiv_{10} 9(32 - 2)$

$$5 + x \equiv_{10} 9(32 - 2)$$

$$5 + x \equiv_{10} 270$$

$$5 + x \equiv_{10} 0$$

$$x \equiv_{10} -5$$

$$x \equiv_{10} 5$$

Solution: $x \equiv_{10} 5$

Solving in Modular Arithmetic

Solve: $7x \equiv_{10} 1$

Solution: $x \equiv_{10} 3$ (Guess and check)

None of our properties so far help us solve this.

There is an algorithm to solve this called the Extended Euclidean Algorithm, if you’re interested.

3 is called the multiplicative inverse of 7 modulo 10, i.e. the value x such that $7x \equiv_{10} 1$.

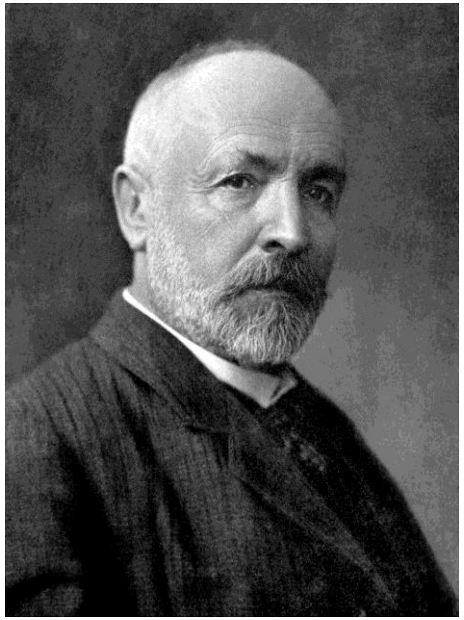
Set Theory

Motivation

Set theory is widely regarded as the foundation for all of mathematics.

In computing, there are applications in:

- Data Structures
- Databases
- Programming Languages



Father of Modern
Set Theory
Georg Cantor
(1845 – 1918)

Sets

Definition:

A **set** is an unordered collection of distinct objects, called elements.

- We write $x \in A$ to say that x is an element of the set A.
- We write $x \notin A$ to say that x is not an element of the set A.

Set Notation

We'll write a set as a collection of elements inside curly braces {}.

Sets are often given variable names with capital letters.

$A = \{0,5,8,10\} = \{5,8,0,10\}$ Sets are unordered

$B = \{\text{watermelon}, \text{apple}, \text{pineapple}\}$ Sets can contain any object

$C = \{a, b, c, c, b, a\} = \{a, b, c\}$ Repeat elements are listed once

$D = \{0,1,2,3,4,5, \dots\}$ Sets can be finite or infinite

Common Sets

$\mathbb{R}$ is the set of Real Numbers. E.g. $1, -17, \pi, \sqrt{2}$

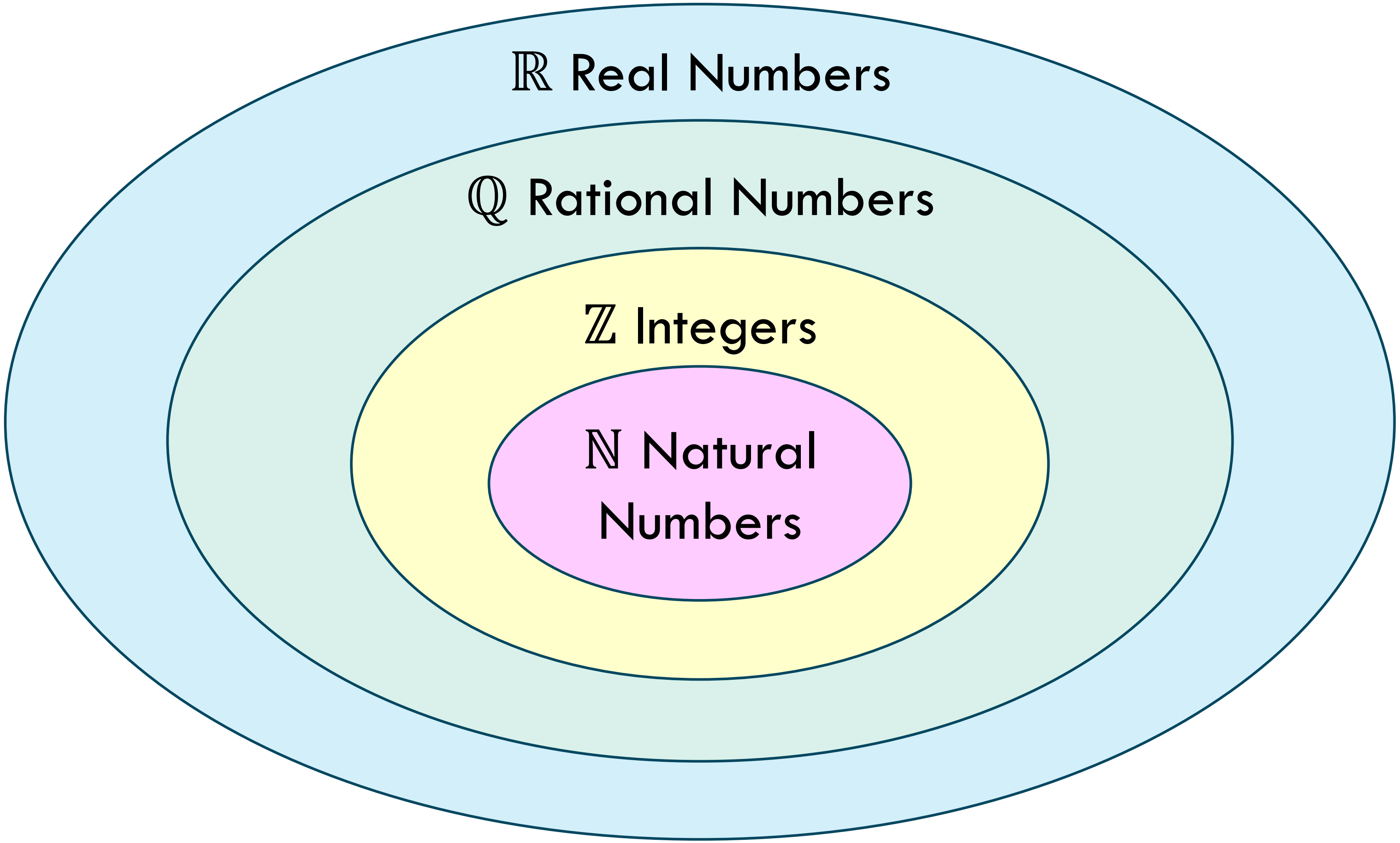
$\mathbb{Z}$ is the set of Integers. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{N}$ is the set of Natural Numbers. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Q}$ is the set of Rational Numbers (fractions) E.g. $\frac{1}{2}, -\frac{11}{3}, 17$

$\emptyset = \{\}$ is the Empty Set $\emptyset$ has no elements

Common Sets


ℝ Real Numbers
ℚ Rational Numbers
ℤ Integers
ℕ Natural
Numbers

Sets can be elements of other sets

For example:

$A = \{\{1\}, \{2\}, \{1,2\}, \emptyset\}$

$B = \{1, 2\}$

Then $1 \in B, 2 \in B$. And $\emptyset \in A, B \in A$.

Sets Builder Notation

Another way to describe a set is using set-builder notation.

$S = \{x : \mathrm{P}(x)\}$ means S is the set of all x for which $\mathrm{P}(x)$ is true.

For example:

- $\{x \in \mathbb{Z} : x > 0\}$ is the set of all positive integers.
- $\{x \in \mathbb{N} : x \equiv_3 2\}$ is the set $\{2, 5, 8, 11, 14, \dots\}$.
- $\left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$ is the set of rational numbers.

Set Cardinality

The cardinality of a set is the number of elements in a set (its size).
The cardinality of a set A is often denoted $|A|$.

What is the cardinality of the following sets?

- $A = \{x \in \mathbb{Z} : x \equiv_4 1 \text{ and } -10 \leq x \leq 10\} = \{-7, -3, 1, 5, 9\}$
  $|\mathrm{A}| = 5$
- $B = \emptyset$
  $|\mathrm{B}| = 0$
- $C = \{\emptyset\}$
  $|C| = 1$

Relationships Between Sets

Set Equality

Sets A and B are equal if they have the same elements.

In predicate logic, $A = B$ is defined as:

$$\forall x\, (x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal?

$C = D = E$

Subset

Set A is a subset of B if every element of A is also in B.

In predicate logic, $A \subseteq B$ is defined as:

$$\forall x(x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

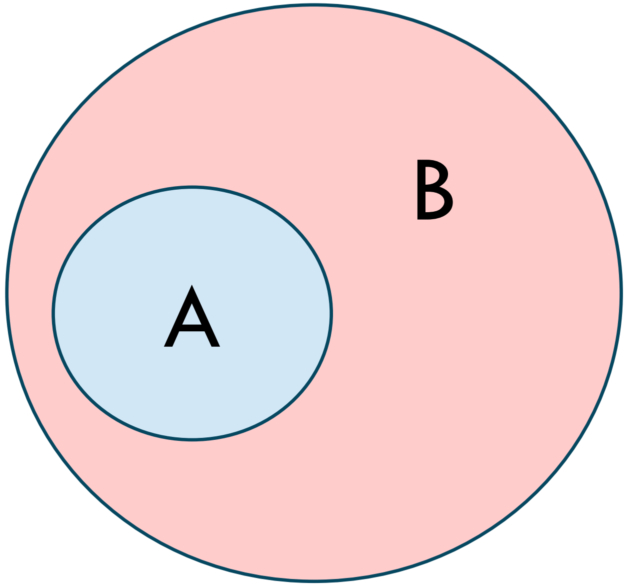
Which sets are subsets?

$C \subseteq B$, $D \subseteq E$, $E \subseteq D$, etc.

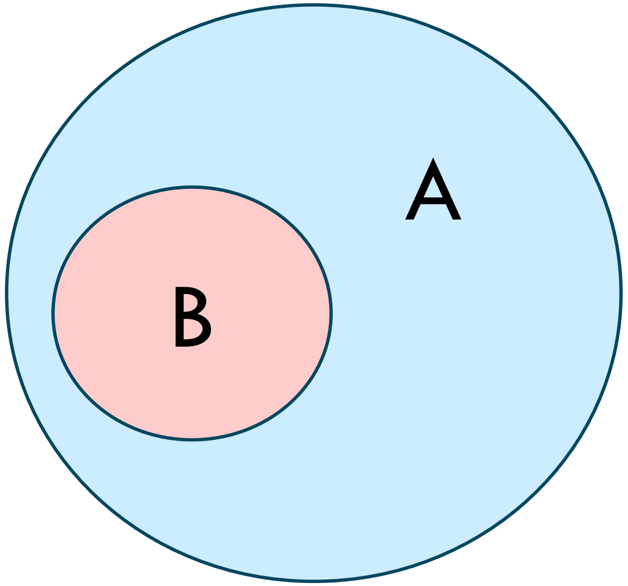
Set Equality and Subsets

$$A = B \equiv A \subseteq B \land B \subseteq A$$

A is a subset of B



B is a subset of A



$\in$ vs. $\subseteq$

$A = \{1, 2, 3\}$ $B = \{2\}$ $C = \{\emptyset, \{2\}\}$

- $\emptyset \subseteq A$? Yes.
- $\emptyset \in A$? No. $\emptyset \in C$ though!
- $2 \subseteq B$? No. $\{2\} \subseteq B$ though!
- $2 \in B$? Yes.
- $B \in A$? No. $B \subseteq A$ though!
- $B \in C$? Yes.

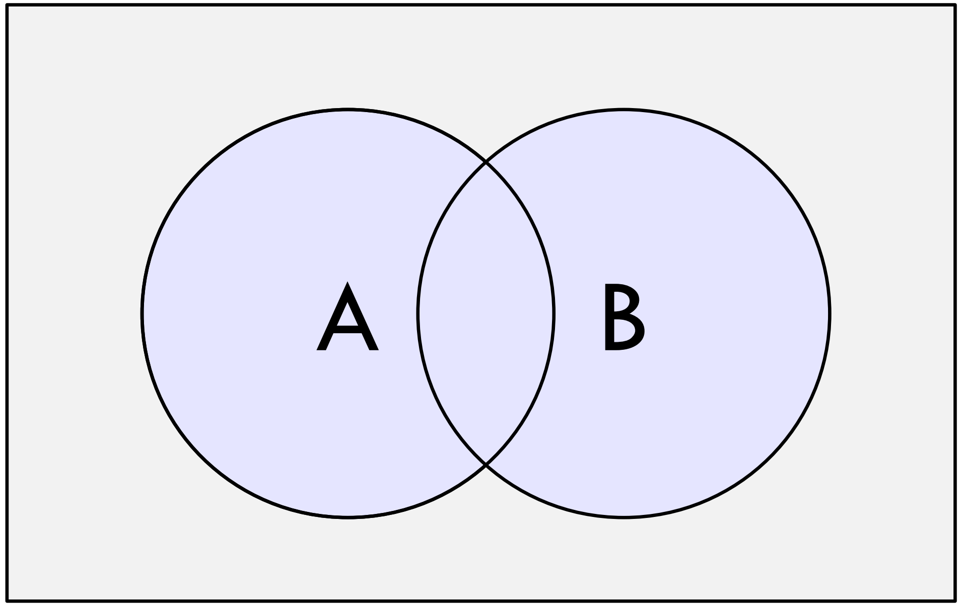
Set Operations

Combining Sets

Set Operations

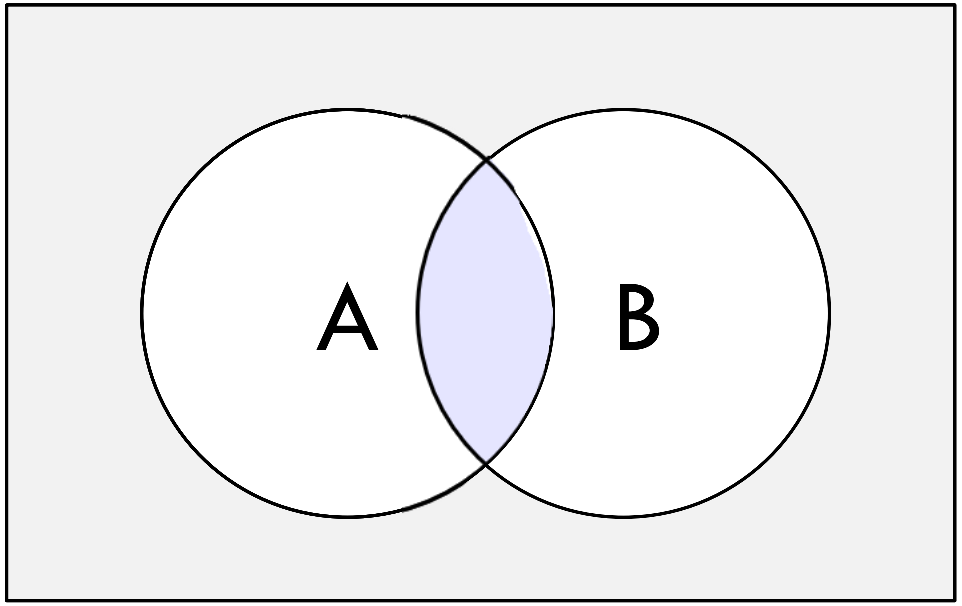
Union: $A \cup B$

$A \cup B = \{x : x \in A \vee x \in B\}$



Intersection: $A \cap B$

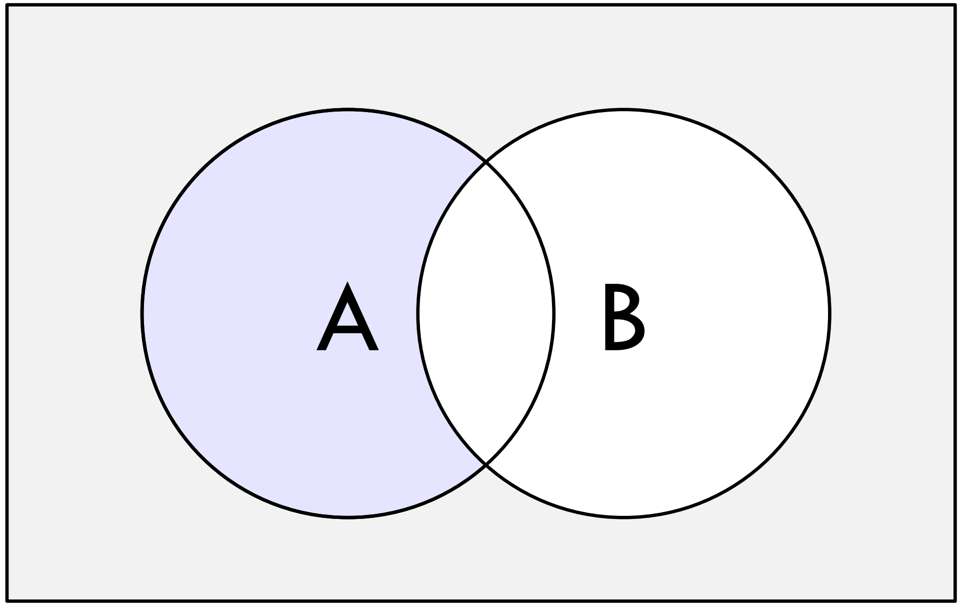
$A \cap B = \{x : x \in A \wedge x \in B\}$



Set Operations

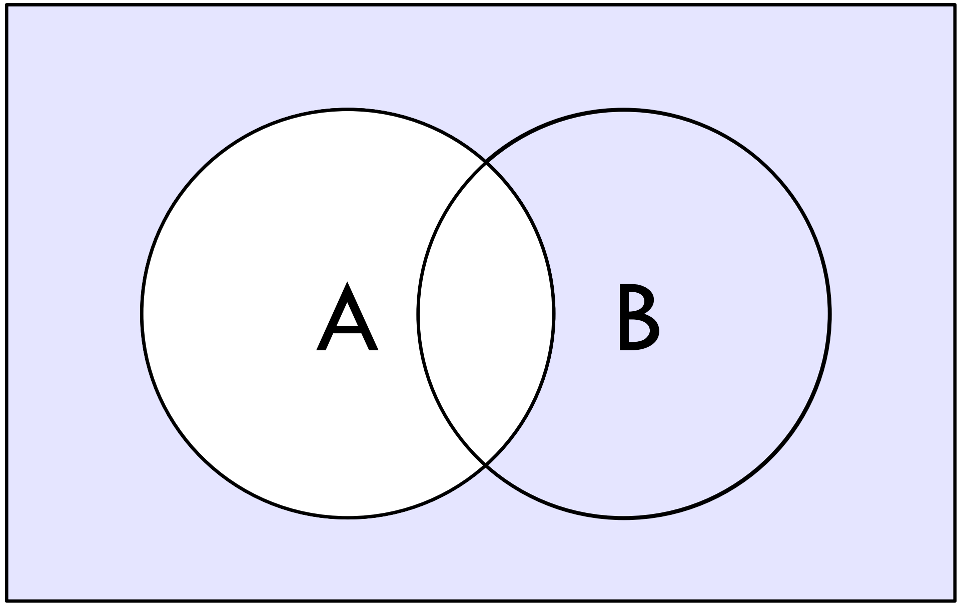
Set Difference: $A \setminus B$

$A \setminus B = \{x : x \in A \wedge x \notin B\}$



Set Complement: $\bar{A} = A^c$
(with respect to the universe $\mathcal{U}$)

$\bar{A} = \{x \in \mathcal{U}\colon\ x \notin A\}$



Set Operations

Erik Brynjolfsson
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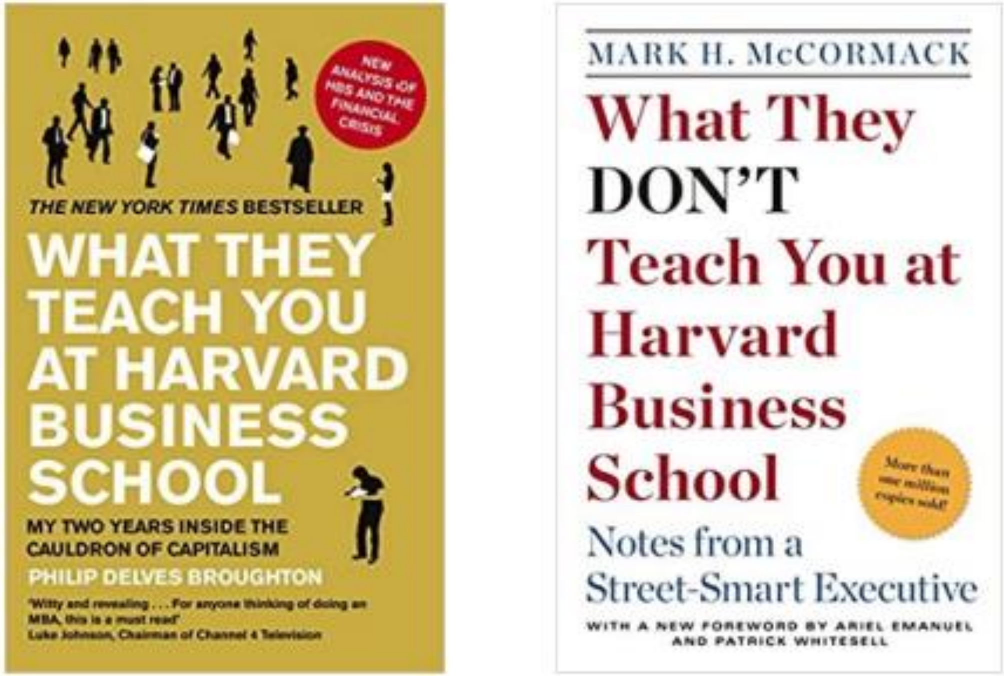
It's remarkable that as recently as 11 years ago, the sum of all human knowledge could be provided in just two books.



Exercises

Definitions

$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$
$A \setminus B = \{x : x \in A \wedge x \notin B\}$
$\bar{A} = \{x : x \notin A\}$

$A = \{1,2,3\}$ $B = \{3,5,6\}$ $C = \{3,4\}$

Using only A, B, C and set operations, make the following sets. The universe is all integers.

- $\{1,2,3,4,5,6\} = A \cup B \cup C$
- $\{3\} = A \cap B$
- $\{1,2\} = A \setminus B = A \cap \bar{B}$

Powerset

Powerset: $\mathcal{P}\{A\}$

$\mathcal{P}(A) = \{X : X \subseteq A\}$

The powerset of A is the **set** of all subsets of A.

$\mathcal{P}(\{1,2\}) = \left\{\emptyset, \{1\}, \{2\}, \{1,2\}\right\}$

$\mathcal{P}(\{a, b, c\}) = \left\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\right\}$

Cartesian Product

Cartesian Product: $A \times B$

$A \times B = \{(a, b) : a \in A, b \in B\}$

The cartesian product of A with B is the set of ordered pairs of the form (a, b), where $a \in A$ and $b \in B$.

If $A = \{1,2\}$ and $B = \{a, b, c\}$ then:

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$\mathbb{R} \times \mathbb{R} =$ the real plane. This is often denoted $\mathbb{R}^2$.

Exercises

Compute the following:

$\{1,2\} \times \emptyset = \emptyset$

$\mathcal{P}(\{2\} \times \{1,3\}) = \{\emptyset, \{(2,1)\}, \{(2,3)\}, \{(2,1),(2,3)\}\}$

$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

$|\mathcal{P}(\{1,2\}) \times \mathcal{P}(\{3,4,5\})| = 32$

Set Proofs

Two Claims

Determine if the following claims are true or false.

Claim 1: For all sets A, B, C, if $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$.

False.

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

True.

Claim 1

Claim 1: For all sets A, B, C, if $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$.

We disprove this claim. Let $A = \{1,2\}$, let $B = \{1\}$ and $C = \{2\}$. Then $A \subseteq (B \cup C)$, but $A \nsubseteq B$ and $A \nsubseteq C$.

Claim 2

Definition
$A \subseteq B \equiv \forall x (x \in A \rightarrow x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof Strategy

- Let A, B, C be arbitrary sets.
- Let $x \in A \cap B \cap C$ be arbitrary.
- Prove that $x \in A \cup B$.

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof

Let sets A, B, C be arbitrary. Let $x \in A \cap B \cap C$ be an arbitrary element. Then by definition of intersection, $x \in A$ and $x \in B$ and $x \in C$. Then certainly $x \in A$. So $x \in A$ or $x \in B$. So by definition of union, $x \in A \cup B$. Since x was arbitrary, $A \cap B \cap C \subseteq A \cup B$.

Anonymous Feedback

https://tinyurl.com/cse311feedback