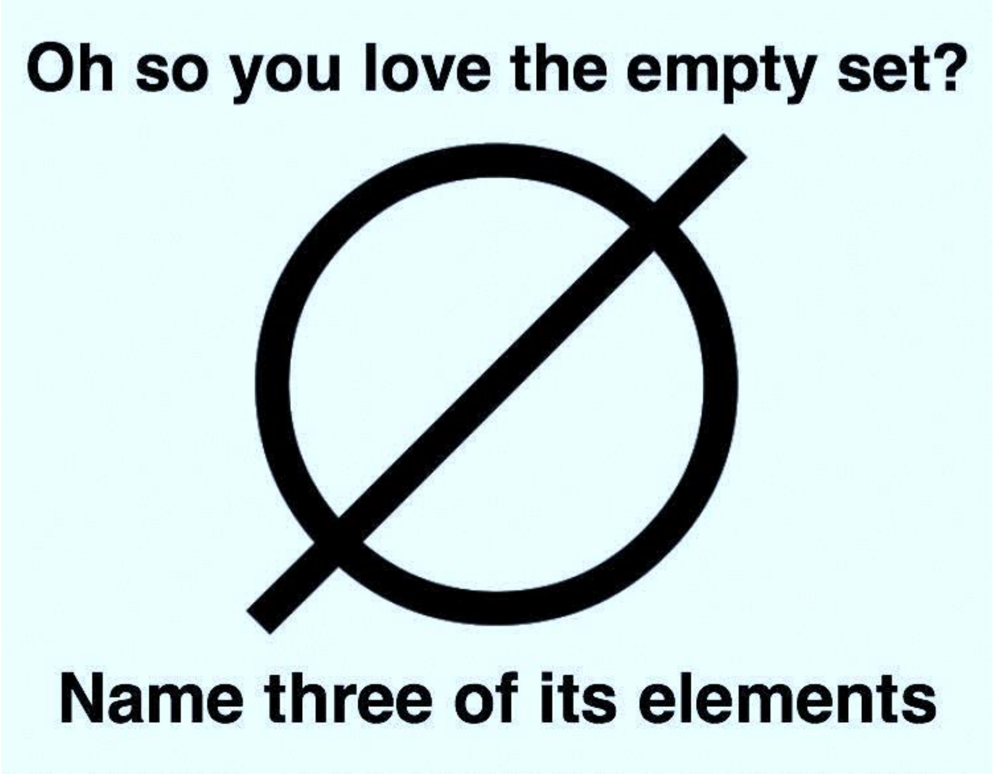


Set Theory

CSE 311: Foundations of Computing I
Lecture 11

Announcements

- HW3 late deadline is today at 11:59 pm
- HW4 has been posted, due 11:59 pm on Wednesday

Wrapping Number Theory

Recall

Let a, b, c, d and $m > 0$ be integers.

- If $a \equiv_m b$, then $b \equiv_m a$.
- If $a \equiv_m b$ and $c \equiv_m d$, then $a + c \equiv_m b + d$.
- If $a \equiv_m b$ and $c \equiv_m d$, then $ac \equiv_m bd$.
- If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.
- $a \equiv_m b$ if and only if $a \,\%\, m = b \,\%\, m$.

Solving in Modular Arithmetic

Solve: $5 + x \equiv_{10} 9(32 - 2)$

$5 + x \equiv_{10} 9(30)$

$5 + x \equiv_{10} \underline{270}$

$5 + x \equiv_{10} \underline{0}$

$x \equiv_{10} -5$

$\boxed{x \equiv_{10} 5}$

Solving in Modular Arithmetic

Solve: $\boxed{7x \equiv_{10} 1}$ $\boxed{x \equiv_{10} 3}$

	0	1	2	3	4	5	6	7	8	9
$7x$	0	7	4	1	8	5	2	9	6	3

Solution: $x \equiv_{10} 3$ (Guess and check)

None of our properties so far help us solve this.

There is an algorithm to solve this called the Extended Euclidean Algorithm, if you're interested.

3 is called the <u>multiplicative inverse</u> of 7 modulo 10, i.e. the value x such that $7x \equiv_{10} 1$.

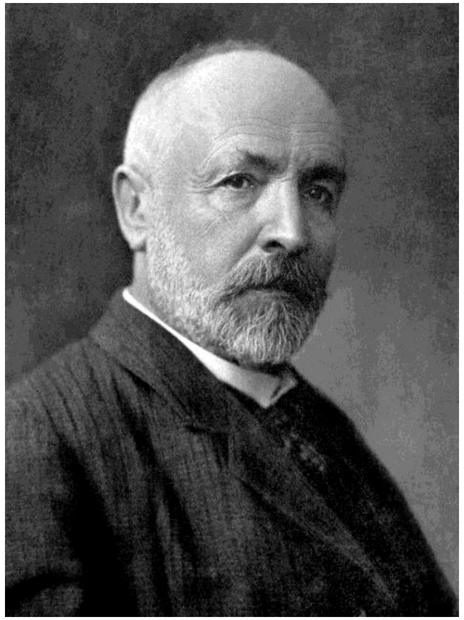
Set Theory

Motivation

Set theory is widely regarded as the foundation for all of mathematics.

In computing, there are applications in:

- Data Structures
- Databases
- Programming Languages



Father of Modern
Set Theory
Georg Cantor
(1845 – 1918)

Sets

Definition:

A **set** is an unordered collection of distinct objects, called elements.

- We write $x \in A$ to say that x is an element of A.
- We write $x \notin A$ to say that x is not an element of A.

$\in \quad \notin$

Set Notation

$E = \{3, a, \text{banana}\}$

We'll write a set as a collection of elements inside curly braces {}.

Sets are often given variable names with capital letters.

$A = \{0, 5, 8, 10\} = \{5, 8, 0, 10\}$

Sets are unordered

$B = \{\text{apple}, \text{cherry}, \text{berry}\}$

Sets can contain any object

$C = \{a, b, c, c, b, a\} = \{a, b, c\}$

Repeat elements are listed once

$D = \{0, 1, 2, 3, \ldots\}$

Sets can be finite or infinite

Common Sets

$\mathbb{R}$ is the set of Real Numbers $1, -17, \sqrt{2}, \pi$

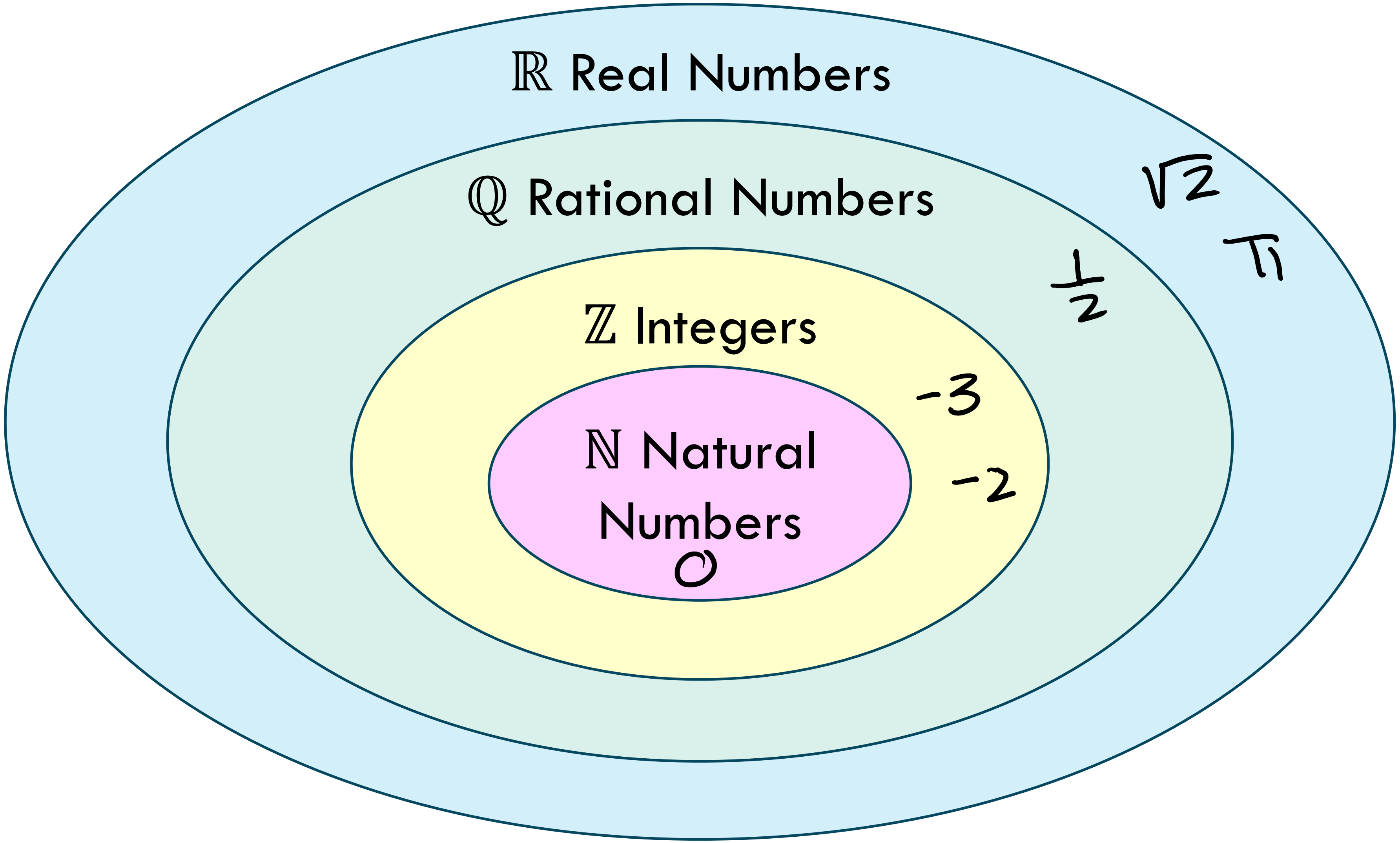
$\mathbb{Z}$ is the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{N}$ is the set of Natural Numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Q}$ is the set of rational numbers (fraction) $\frac{1}{2}, \frac{-11}{3}, 17$

$\emptyset$ is the empty set $\{\} = \emptyset$

Common Sets


ℝ Real Numbers
ℚ Rational Numbers
ℤ Integers
ℕ Natural
Numbers
0
-3
-2
1/2
√2
π

Sets can be elements of other sets

Set<Set<Integer>>

For example:

$A = \{\{1\}, \{2\}, \{1,2\}, \emptyset\}$

$B = \{1, 2\}$

Then $1 \in B$, $2 \in B$. And $\{1\} \in A$, $B \in A$, $\emptyset \in A$.

Sets Builder Notation

Another way to describe a set is using set-builder notation.

$S = \{x : \mathrm{P}(x)\}$ means S is the set of all x for which P(x) is true.

For example:

- $\{x \in \mathbb{Z} : x > 0\}$ is the set of positive integers.
- $\{x \in \mathbb{N} : x \equiv_3 2\}$ is the set $\{2, 5, 8, 11, 14, \ldots\}$.
- $\left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$ is the set of rational numbers $\mathbb{Q}$.

Set Cardinality

The cardinality of a set is the number of elements in the set.

The cardinality of a set A is often denoted $|A|$.

What is the cardinality of the following sets?

- $A = \{x \in \mathbb{Z} : x \equiv_4 1 \text{ and } -10 \leq x \leq 10\}$ $|A| = 5$

  $-7, -3, 1, 5, 9$

- $B = \emptyset$ $|B| = 0$

- $C = \{\emptyset\}$ $|C| = 1$

Relationships Between Sets

Set Equality

Sets A and B are **equal** if they have the same element.

In predicate logic, $A = B$ is defined as:

$$A = B \equiv \forall x (x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

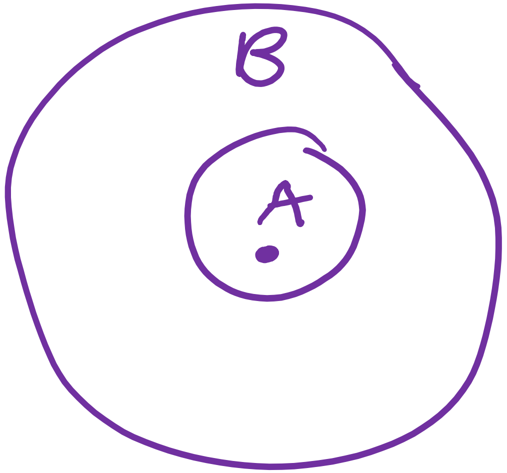
Which sets are equal? $C = D = E$

Subset

Set A is a **subset** of B if every element of A is an element of B.

In predicate logic, $A \subseteq B$ is defined as:

$$\forall x\,(x \in A \rightarrow x \in B)$$



A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are subsets?

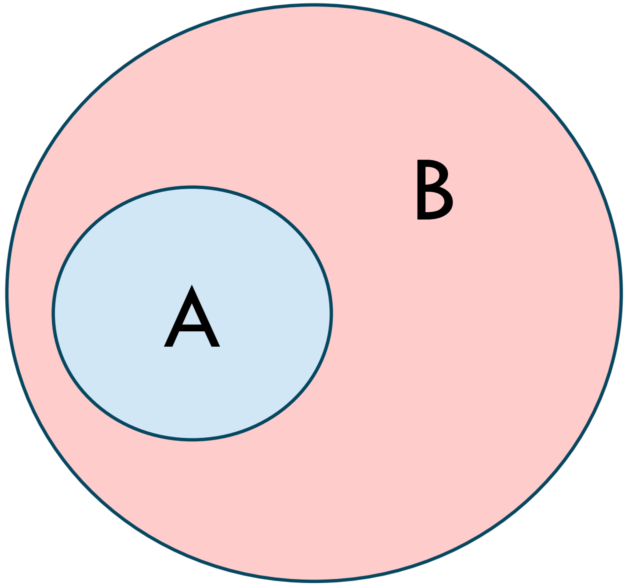
$A \subseteq A$

$C \subseteq B$ $D \subseteq B$ $E \subseteq B$ $C \subseteq D$ $D \subseteq C$

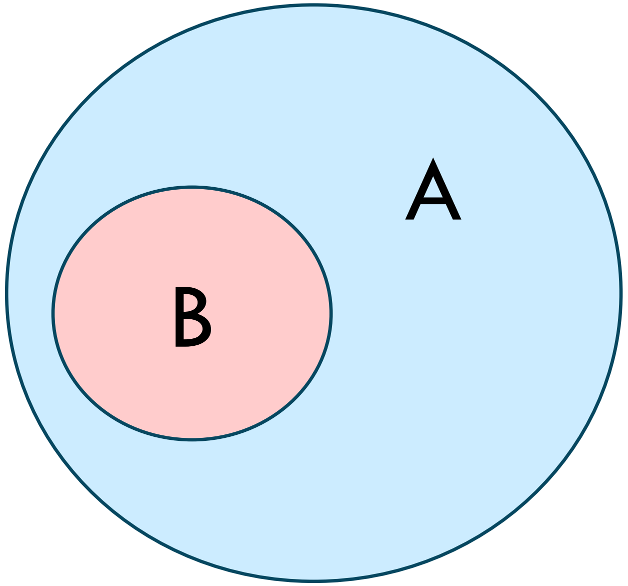
Set Equality and Subsets

$$A = B \quad \equiv \quad A \subseteq B \wedge B \subseteq A$$

A is a subset of B



B is a subset of A



∈ vs. ⊆

$A = \{1, 2, 3\}$ $B = \{2\}$ $C = \{\emptyset, \{2\}\}$

$1 \in A$

$\{1\} \subseteq A$

- $\emptyset \subseteq A$? Yes $\{\} \subseteq \{1, 2, 3\}$ vacuously
- $\emptyset \in A$? No $\emptyset \in C$ though
- $2 \subseteq B$? No
- $2 \in B$? Yes
- $B \in A$? No $B \subseteq A$ though
- $B \in C$? Yes

Set Operations

Combining Sets

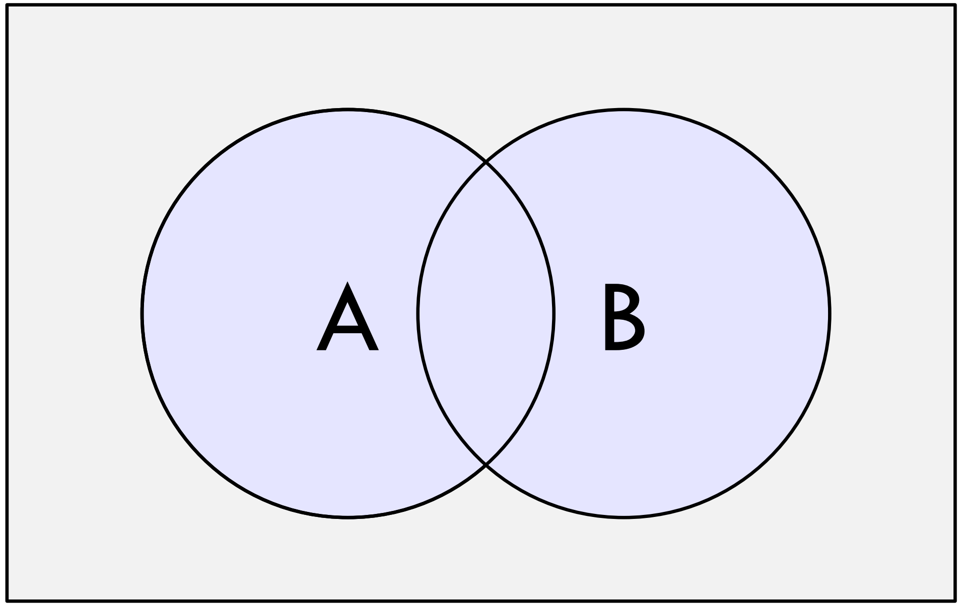
$A = \{a, b, c\}$ $\quad A \cup B = \{a, b, c, d\}$

$B = \{c, d\}$ $\quad A \cap B = \{c\}$

Set Operations

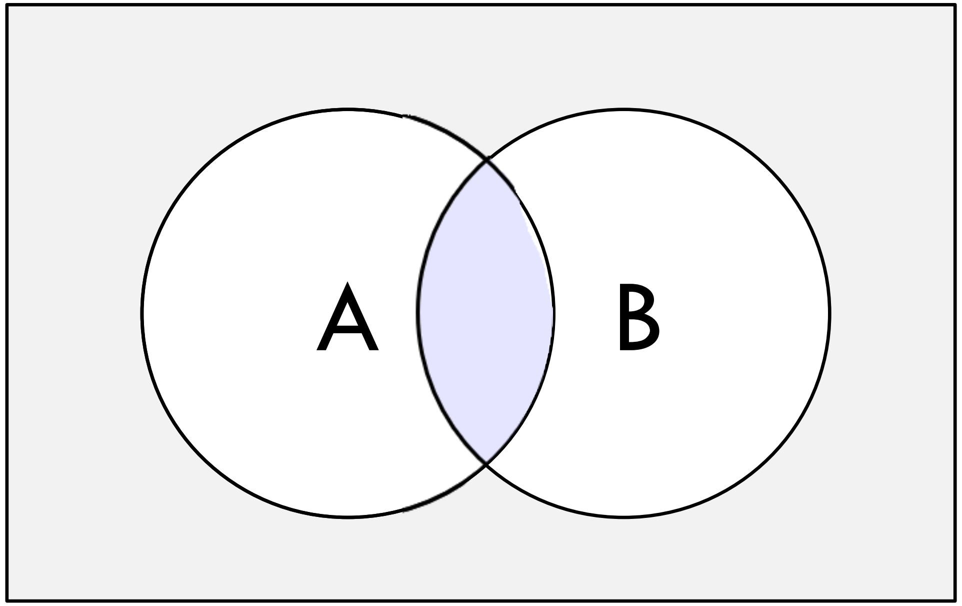
Union: $A \cup B$

$A \cup B = \{x : x \in A \vee x \in B\}$



Intersection: $A \cap B$

$A \cap B = \{x : x \in A \wedge x \in B\}$



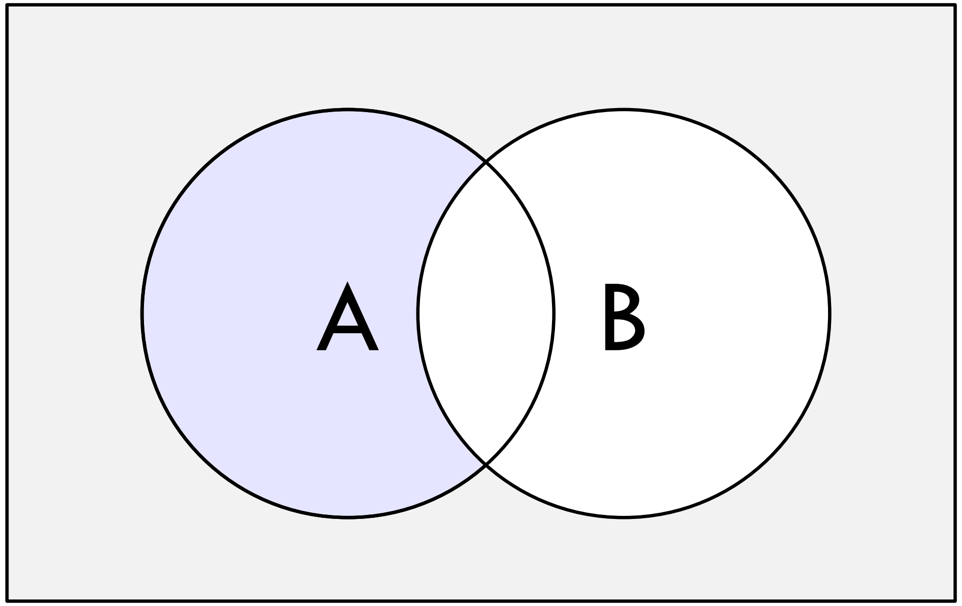
Set Operations

$A = \{a, b, c\}$ $A \setminus B = \{a, b\}$

$B = \{c, d\}$ $\bar{A} = \{e, f, \ldots, z\}$

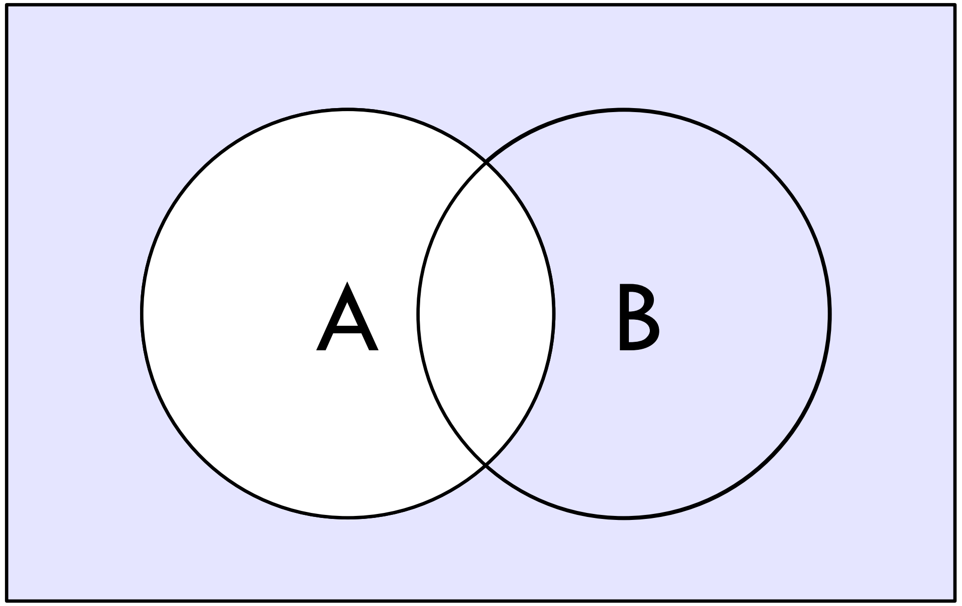
Set Difference: $A \setminus B$

$A \setminus B = \{x : x \in A \land x \notin B\}$



Set Complement: $\bar{A} = A^c$
(with respect to the universe $\mathcal{U}$)

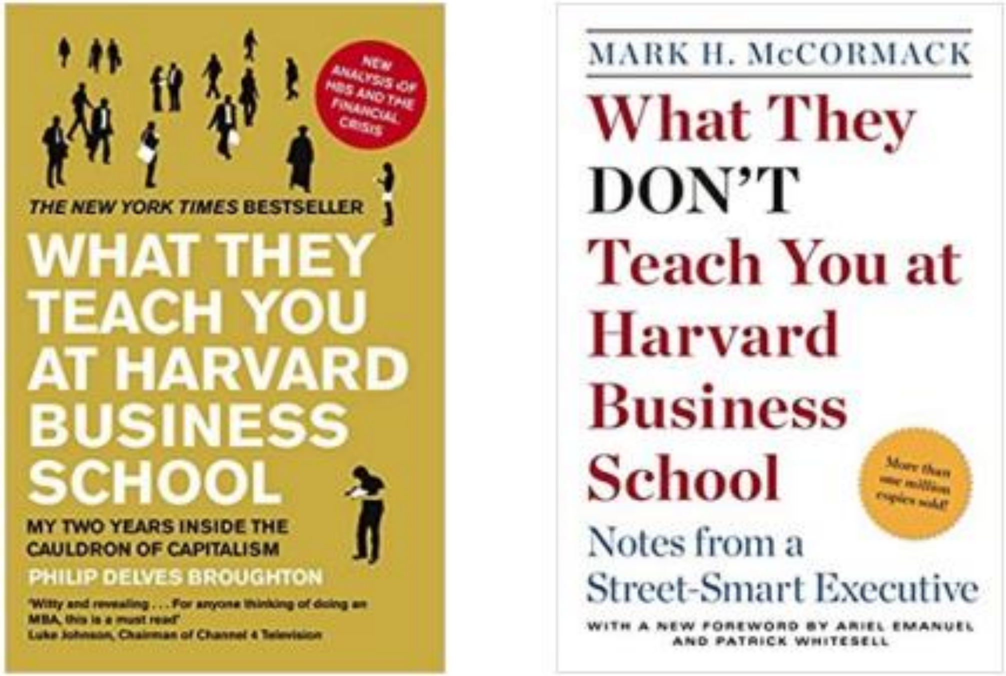
$\bar{A} = \{x \in \mathcal{U}:\ x \notin A\}$



Set Operations

Erik Brynjolfsson
@erikbryn

It's remarkable that as recently as 11 years ago, the sum of all human knowledge could be provided in just two books.



Exercises

Definitions

$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$
$A \setminus B = \{x : x \in A \wedge x \notin B\}$
$\bar{A} = \{x : x \notin A\}$

$A = \{1, 2, 3\}$ $\quad B = \{3, 5, 6\}$ $\quad C = \{3, 4\}$

Using only A, B, C and set operations, make the following sets. The universe is all integers.

- $\{1, 2, 3, 4, 5, 6\}$ $\quad A \cup B \cup C$
- $\{3\}$ $\quad A \cap C = A \cap B$
- $\{1, 2\}$ $\quad A \setminus B = A \setminus C = \bar{C} \cap A = \bar{B} \cap A$
  See if you can do it without using set difference! $\quad \overline{(B \cup C)}$?

Powerset

Powerset: $\mathcal{P}\{A\}$

2^n

$$\mathcal{P}(A) = \{X : X \subseteq A\}$$

The powerset of A is the **set** of all subsets of A.

$\checkmark$ 2

$\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ 4

$\mathcal{P}(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$ 8

3

Cartesian Product

Cartesian Product: $A \times B$

$A \times B = \{(a, b) : \underline{a \in A}, \underline{b \in B}\}$

The cartesian product of A with B is the set of ordered pairs of the form (a, b), where $a \in A$ and $b \in B$.

$\underline{(a, 1)} \in B \times A$

$(a, 1) \notin A \times B$

If $A = \{1,2\}$ and $B = \{a, b, c\}$ then:

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$\mathbb{R} \times \mathbb{R} =$ the real plane $\mathbb{R}^2$

Exercises

Compute the following:

$\{1,2\} \times \emptyset = \emptyset$

$\mathcal{P}(\{2\} \times \{1,3\}) = \mathcal{P}(\{(2,1),(2,3)\})$

$= \{\emptyset, \{(2,1)\}, \{(2,3)\}, \{(2,1),(2,3)\}\}$

$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

$|\mathcal{P}(\underbrace{\{1,2\}}_{4}) \times \underbrace{\mathcal{P}(\{3,4,5\})}_{8}| = 32$

Set Proofs

Two Claims

Determine if the following claims are true or false.

Claim 1: For all sets A, B, C, if $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$.

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Claim 1

Claim 1: For all sets A, B, C, if $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$.

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \to x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof Strategy

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof

Anonymous Feedback

https://tinyurl.com/cse311feedback