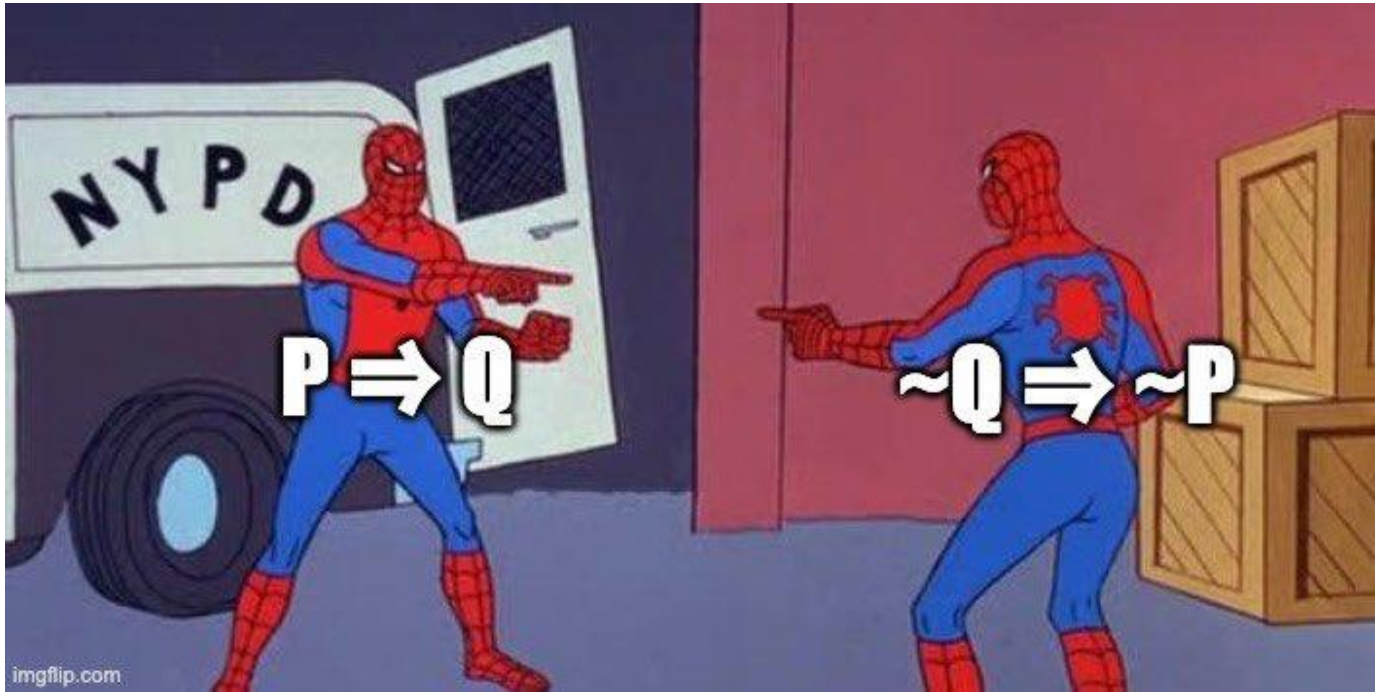

Proof by Contrapositive, Proof of Biconditional

CSE 311: Foundations of Computing I
Lecture 7

Announcements

- HW1 grades posted on Gradescope, printed copies of the solutions are at the front
- HW2 is due tonight at 11:59 pm

Review: Direct Proof

Direct Proof

Direct proof is one strategy for proving statements of the form
$\forall x\,\big(\mathrm{P}(x) \to \mathrm{Q}(x)\big)$.

It involves:

- Taking an arbitrary x in the domain
- Assuming $\mathrm{P}(x)$ is true
- Proving that $\mathrm{Q}(x)$ is true

Direct Proof Example

<u>Definitions</u>
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \Big(\big(\mathrm{Odd}(x) \land \mathrm{Odd}(y)\big) \to \mathrm{Odd}(xy) \Big)$$

Let x and y be arbitrary integers. Suppose that x and y are odd. Then by definition of odd, there exists some integer k such that $x = 2k + 1$, and some integer j such that $y = 2j + 1$.

Then multiplying x and y, we can see that:

$$xy = (2k + 1) \cdot (2j + 1) = 4kj + 2j + 2k + 1 = 2(2kj + j + k) + 1$$

Since k, j are integers, $2kj + j + k$ is an integer. So xy is 2 times an integer plus 1. So by definition of odd, xy is odd.

Since x, y were arbitrary, we have shown that the product of two odd integers is odd.

Proof Strategy: Contrapositive

Proof by Contrapositive

Proof by contrapositive is another strategy for proving statements of the form $\forall x\big(\mathrm{P}(x) \rightarrow \mathrm{Q}(x)\big)$.

The strategy is to prove the contrapositive, i.e. prove $\forall x\big(\neg\mathrm{Q}(x) \rightarrow \neg\mathrm{P}(x)\big)$

Proof by Contrapositive

Definitions
$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

What's the claim in logic? $\forall x\big(\text{Odd}(3x + 2) \rightarrow \text{Odd}(x)\big)$

Try to prove this claim with a direct proof.

Let x be an arbitrary integer. Suppose that $3x + 2$ is odd. Then $3x + 2 = 2k + 1$ for some integer k. Subtracting both sides by 2, we have $3x = 2k - 1$. Then $x = \frac{2k-1}{3}$...?

Note: We actually *can* prove this directly. But it's much less obvious to see.

Proof by Contrapositive

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

$$\forall x\big(\mathrm{Odd}(3x+2) \rightarrow \mathrm{Odd}(x)\big) \equiv \forall x\big(\mathrm{Even}(x) \rightarrow \mathrm{Even}(3x+2)\big)$$

We prove by contrapositive. Let x be an arbitrary integer. Suppose that x is even. Then by definition of even, $x = 2k$ for some integer k. Consider $3x + 2$:

$$3x + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

Since k is an integer, $3k + 1$ is an integer. So $3x + 2$ is 2 times an integer. So by definition of even, $3x + 2$ is even. Since x was arbitrary, we have shown that for all integers x, if x is even then $3x + 2$ is even. Thus the contrapositive also holds: for all integers x, if $3x + 2$ is odd, then x is odd.

Proof by Contrapositive

How do we identify *when* to use a direct proof vs. a proof by contrapositive?

Try a direct proof first. If it seems challenging, then consider the contrapositive.

Another Proof by Contrapositive

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove by Contrapositive: For an integer n, if n^3 is even, then n is even.

$$\forall n\left(\text{Even}(n^3) \to \text{Even}(n)\right) \equiv \forall n\left(\text{Odd}(n) \to \text{Odd}(n^3)\right)$$

We prove by contrapositive. Let n be an arbitrary integer. Suppose that n is odd. Then by definition of odd, $n = 2k + 1$ for some integer k. Consider n^3:

$$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2\left(4k^3 + 6k^2 + 3k\right) + 1$$

Since k is an integer, $4k^3 + 6k^2 + 3k$ is an integer. Thus by definition of odd, n^3 is odd. Since n was arbitrary, we have shown that for all integers n, if n is odd then n^3 is odd. Thus the contrapositive also holds: for all integers n, if n^3 is even, then n is even.

Remark: Proof by Contrapositive

Just like we can show $p \to q$ is true by using a direct proof of $\neg q \to \neg p$, we can use our other logical equivalences.

Suppose for example the original claim is of the form $p \to (q \vee r)$. Then the contrapositive would be:

$$\begin{aligned} p \to (q \vee r) &\equiv \neg(q \vee r) \to \neg p \\ &\equiv (\neg q \wedge \neg r) \to \neg p \end{aligned}$$

So the proof by contrapositive would be of the form:

Suppose $\neg q$ and $\neg r$. Try to show $\neg p$.

Proof Strategy: Biconditional

Proof of a Biconditional

Recall that biconditionals are statements of the form:
$\forall x\big(\mathrm{P}(x) \leftrightarrow \mathrm{Q}(x)\big)$

The strategy is to prove such statements is to prove an implication in both directions. I.e. prove $\forall x\big(\mathrm{P}(x) \rightarrow \mathrm{Q}(x)\big) \land \forall x\big(\mathrm{Q}(x) \rightarrow \mathrm{P}(x)\big)$.

Proof of a Biconditional

Prove: For an integer x, $2x + 3 = 15$ if and only if $x = 6$.

$\Rightarrow$ Let x be an arbitrary integer. Suppose $2x + 3 = 15$. Then $2x = 12$. Then $x = 6$. Since x was arbitrary, for all integers x if $2x + 3 = 15$ then $x = 6$.

$\Leftarrow$ Let x be an arbitrary integer. Suppose $x = 6$. Then consider $2x + 3$:

$$2x + 3 = 2(6) + 3 = 12 + 3 = 15$$

Since x was arbitrary, for all integers x if $x = 6$ then $2x + 3 = 15$.

Remark: Biconditional Proofs

Each direction of the biconditional proof can use whichever proof type fits best (direct, contrapositive, etc.).

Consider the claim: For an integer n, $3n + 3$ is odd iff n is even.

$\Leftarrow$ Prove that $\forall n(\text{Even}(n) \rightarrow \text{Odd}(3n+3))$. Use direct proof.

$\Rightarrow$ Prove that $\forall n(\text{Odd}(3n+3) \rightarrow \text{Even}(n))$. Use contrapositive. I.e. prove that $\forall n(\text{Odd}(n) \rightarrow \text{Even}(3n+3))$.

Another Proof of a Biconditional

Definitions
$\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer n, $3n + 3$ is odd iff n is even.

$\Leftarrow$ Let n be an arbitrary integer. Suppose n is even. Then by definition of even, $n = 2k$ for some integer k. Then consider $3n + 3$:

$$3n + 3 = 3(2k) + 3 = 6k + 3 = 2(3k + 1) + 1$$

Since k is an integer, $3k + 1$ is an integer. So $3n + 3$ is 2 times an integer plus 1. So by definition of odd, $3n + 3$ is odd. Since n was arbitrary, this shows that for all integers n if n is even then $3n + 3$ is odd.

$\Rightarrow$ We prove by contrapositive. Let n be an arbitrary integer. Suppose that n is odd. Then by definition of odd, $n = 2k + 1$ for some integer k. Then consider $3n + 3$:

$$3n + 3 = 3(2k + 1) + 3 = 6k + 3 + 3 = 6k + 6 = 2(3k + 3)$$

Since k is an integer, $3k + 3$ is an integer. So $3n + 3$ is 2 times an integer. So by definition of even, $3n + 3$ is even. Since n was arbitrary, this shows that for all integers n, if n is odd then $3n + 3$ is even. Then the contrapositive also holds: for all integers n, if $3n + 3$ is odd then n is even.

Remark: Multiple Biconditionals

Suppose you wanted to prove $p \leftrightarrow q \leftrightarrow r$.

How many sub-proofs would you need?

Could do every pair: $(p \rightarrow q) \wedge (q \rightarrow p) \wedge (q \rightarrow r) \wedge (r \rightarrow q) \wedge (p \rightarrow r) \wedge (r \rightarrow p)$

But it turns out you only need 3. For instance, $(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)$

Any chain of conditional statements work so long as you can follow the chain of implications to get from any statement to any other.

Proof Strategies So Far

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Proof of Existence
- Disproof

Material for HW3 will be finished on Friday's lecture. The assignment will be posted tonight so that you can get started.