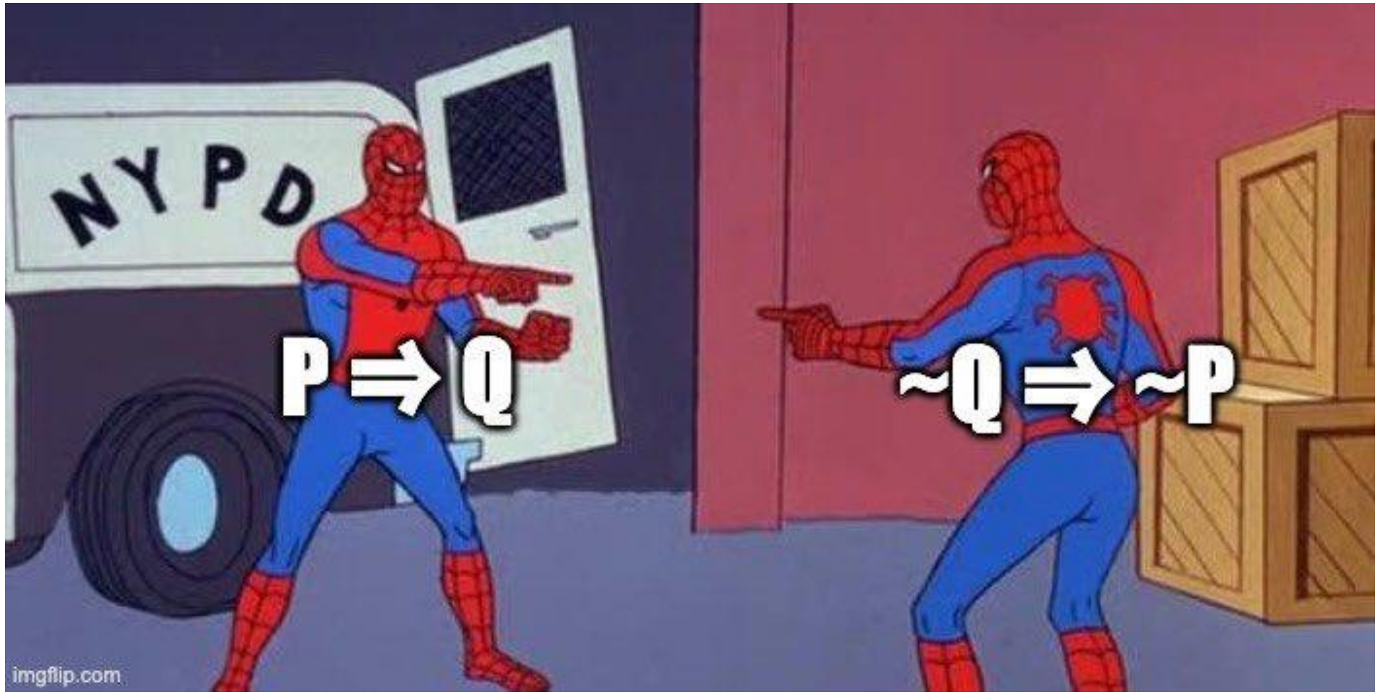

Proof by Contrapositive, Proof of Biconditional

CSE 311: Foundations of Computing I
Lecture 7

Announcements

- HW1 grades posted on Gradescope, printed copies of the solutions are at the front
- HW2 is due tonight at 11:59 pm

Review: Direct Proof

Direct Proof

Direct proof is one strategy for proving statements of the form $\forall x\,(\mathrm{P}(x) \to \mathrm{Q}(x))$.

It involves:

- Taking an arbitrary x in the domain
- Assuming $\mathrm{P}(x)$ is true
- Proving that $\mathrm{Q}(x)$ is true

Direct Proof Example

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \left(\left(\mathrm{Odd}(x) \land \mathrm{Odd}(y) \right) \rightarrow \mathrm{Odd}(xy) \right)$$

Let x and y be arbitrary integers. Suppose that x and y are odd. Then by definition of odd, there exists some integer k such that $x = 2k + 1$, and some integer j such that $y = 2j + 1$.

Then multiplying x and y, we can see that:

$$xy = (2k + 1) \cdot (2j + 1) = 4kj + 2j + 2k + 1 = 2(2kj + j + k) + 1$$

Since k, j are integers, $2kj + j + k$ is an integer. So xy is 2 times an integer plus 1. So by definition of odd, xy is odd.

Since x, y were arbitrary, we have shown that the product of two odd integers is odd.

Proof Strategy: Contrapositive

Proof by Contrapositive

$P \to Q \equiv \neg Q \to \neg P$

Proof by contrapositive is another strategy for proving statements of the form $\forall x\,(P(x) \to Q(x))$.

The strategy is to $\forall x\,(\neg Q(x) \to \neg P(x))$.

Proof by Contrapositive

Definitions
$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

What's the claim in logic? $\forall x(\text{Odd}(3x+2) \to \text{Odd}(x))$

Try to prove this claim with a direct proof.

Let x be an arbitrary integer. Suppose that $3x+2$ is odd. By def of odd, there exists some integer k such that $3x+2 = 2k+1$

$$3x+2 = 2k+1$$
$$3x = 2k-1$$
$$x = \frac{2k-1}{3} \quad ...?$$

Proof by Contrapositive

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

$\forall x\,(\mathrm{Even}(x) \to \mathrm{Even}(3x+2))$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd. $\forall x(\mathrm{Odd}(3x + 2) \to \mathrm{Odd}(x))$

We prove by contrapositive. Let x be an arbitrary integer. Suppose that x is even. Then by def. of even there exists some integer k such that $x = 2k$. Consider $3x+2$:

$$3x+2 = 3(2k)+2 = 6k+2 = 2(3k+1)$$

Since k is an integer, $3k+1$ is an integer. So by def. of even, $3x+2$ is even. Thus since x was arbitrary, for all integers x, if x is even then $3x+2$ is even. So the contrapositive also holds, i.e. for all integers x, if $3x+2$ is odd then x is odd.

Proof by Contrapositive

How do we identify *when* to use a direct proof vs. a proof by contrapositive?

Try a direct proof first. If it seems challenging, then consider the contrapositive.

Another Proof by Contrapositive

Definitions
$\text{Even}(x) := \exists k(x = 2k)$

$\forall n(\text{Even}(n^3) \to \text{Even}(n)) \equiv \forall n(\text{Odd}(n) \to \text{Odd}(n^3))$

Prove by Contrapositive: For an integer n, if n^3 is even, then n is even.

We prove by contrapositive. Let n be an arbitrary integer. Suppose that n is odd. By definition of odd, $n = 2k+1$ for some integer k. Consider n^3:

$n^3 = (2k+1)^3 = (2k+1)(2k+1)^2 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$

Since k is an integer, $4k^3 + 6k^2 + 3k$ is an integer. So by definition of odd, n^3 is odd. Since n was arbitrary, for all integers n, if n is odd, n^3 is odd. Thus the contrapositive also holds.

Remark: Proof by Contrapositive

Just like we can show $p \to q$ is true by using a direct proof of $\neg q \to \neg p$, we can use our other logical equivalences.

Suppose for example the original claim is of the form $p \to (q \vee r)$. Then the contrapositive would be:

$$p \to (q \vee r) \equiv \neg(q \vee r) \to \neg p$$
$$\equiv (\neg q \wedge \neg r) \to \neg p$$

So the proof by contrapositive would be of the form:

suppose $\neg q$ and $\neg r$ ... show $\neg p$.

Proof Strategy: Biconditional

Proof of a Biconditional

$p \leftrightarrow q \equiv p \to q \wedge q \to p$

Recall that biconditionals are statements of the form:

$\forall x\, (P(x) \leftrightarrow Q(x))$ ________.

The strategy is to prove such statements is to prove an implication

, in both directions. ________.

$\forall x\, (P(x) \to Q(x)) \wedge \forall x\, (Q(x) \to P(x))$

Proof of a Biconditional

$\forall x\,(2x+3=15 \leftrightarrow x=6)$

$2x+3=15$

$2x=12$

$x=6$

$x=5$

$x^2=25$

Prove: For an integer x, $2x+3=15$ if and only if $x=6$.

$\Rightarrow$ Let x be arbitrary. Suppose $2x+3=15$. Then subtracting 3, $2x=12$. Dividing by 2, $x=6$. Since x was arbitrary, for all integers x, if $2x+3=15$ then $x=6$.

$\Leftarrow$ Let x be arbitrary. Suppose $x=6$. Then multiplying by 2, $2x=12$. Adding 3, $2x+3=15$. Since x was arbitrary, for all integers x, if $x=6$, then $2x+3=15$.

Remark: Biconditional Proofs

Each direction of the biconditional proof can use whichever proof type fits best (direct, contrapositive, etc.).

Consider the claim: For an integer n, $3n + 3$ is odd iff n is even.

$\Rightarrow$ $\forall n\,(\text{Odd}(3n+3) \rightarrow \text{Even}(n))$ Contrapositive

i.e. $\forall n\,(\text{Odd}(n) \rightarrow \text{Even}(3n+3))$

$\Leftarrow$ $\forall n\,(\text{Even}(n) \rightarrow \text{Odd}(3n+3))$ Direct

Another Proof of a Biconditional

Definitions
$\text{Even}(x) := \exists k(x = 2k)$
$\text{Odd}(x) := \exists k(x = 2k + 1)$

Prove: For an integer n, $3n + 3$ is odd iff n is even.

$\Rightarrow$ We prove by contrapositive. Suppose n is odd. Then $n = 2k+1$ for some integer k by definition of odd. Consider $3n+3$:

$$3n+3 = 3(2k+1) + 3 = 6k + 3 + 3 = 6k+6 = 2(3k+3)$$

Since k is an integer, $3k+3$ is an integer. So by def. of even, $3n+3$ is even. Since n was arbitrary, for all integers n, if n is odd then $3n+3$ is even. Thus the contrapositive also holds.

$\Leftarrow$ Suppose n is even. Then $n = 2k$ for some integer k. Then consider $3n+3$:

$$3n+3 = 3(2k)+3 = 6k+3 = 2(3k+1)+1$$

Since k is an integer, $3k+1$ is an integer. So $3n+3$ is odd by def of odd. Since n was arbitrary, for all integers n if n is even then $3n+3$ is odd.

Another Proof of a Biconditional

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$
$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer n, $3n + 3$ is odd iff n is even.

Prove directly instead

$\Rightarrow$ Let n be an arbitrary integer. Suppose $3n+3$ is odd. Then by def of odd, there exists some integer k such that $3n+3 = 2k+1$.

$$3n+3 = 2k+1$$
$$3n = 2k-2$$
$$n = 2k-2-2n$$
$$n = 2(k-1-n)$$

Since k and n are integers, $k-1-n$ is an integer. So n is even by def of even. Since n was arbitrary, claim holds.

Remark: Multiple Biconditionals

Suppose you wanted to prove $p \leftrightarrow q \leftrightarrow r$. $\forall x\,(P(x) \leftrightarrow Q(x) \leftrightarrow R(x))$

How many sub-proofs would you need?   Only 3!

6: $p \to q$, $q \to p$, $p \to r$, $r \to p$, $q \to r$, $r \to q$

4: $p \to q$, $q \to p$, $q \to r$, $r \to q$

3: $p \to q$, $q \to r$, $r \to p$

Proof Strategies So Far

- Direct Proof
- Proof by Contrapositive $\forall$
- Proof of Biconditional
- Proof by Cases
- Proof of Existence $\exists$
- Disproof

Material for HW3 will be finished on Friday's lecture. The assignment will be posted tonight so that you can get started.