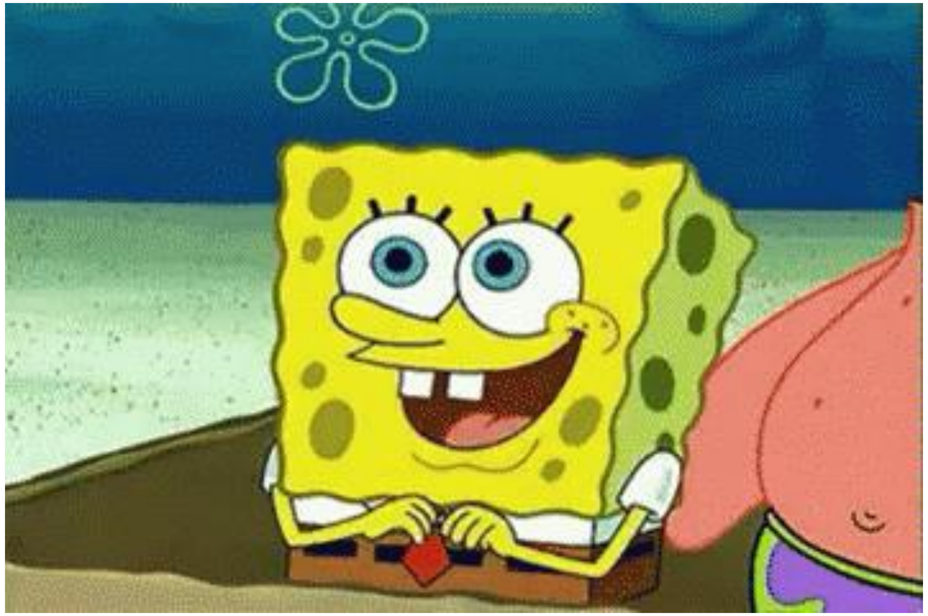

Direct Proof

CSE 311: Foundations of Computing I
Lecture 6

Announcements

- HW1 solutions at the front
- HW2 due Wednesday, 11:59 pm
- Thank you for your feedback!

Review: Predicate Logic

<u>3 Parts</u>

1. Predicate – Function that outputs true or false.
   $\mathrm{Prime}(x) \coloneqq x$ is prime

2. Domain of Discourse – Set of possible inputs to a predicate.
   E.g. Integers

3. Quantifiers – A statement about when a predicate is true
   For all: ∀   There exists: ∃

Theorems and Proofs

Theorems and Proofs

Definition:

A **theorem** is ____.

A **proof** is ____.

Theorems and Proofs

Examples of theorems include…

- Given a right triangle with side lengths a, b and hypotenuse c, $a^2 + b^2 = c^2$
- There are infinitely many prime numbers.
- There exists a problem that cannot be solved by a program.

Integer

Definition:

An **integer** is ____.

Odd and Even

Definitions:

An integer x is even iff ____.

An integer x is odd iff ____.

Proof Strategy: Direct Proof

Direct Proof

Direct proof is one strategy for proving statements of the form

____.

Our First Direct Proof

Prove: “For all integers x, if x is even, then x^2 is even.”

What’s the claim in logic?

How would we prove this claim?

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x \left(\text{Even}(x) \rightarrow \text{Even}(x^2)\right)$

Direct Proof Template

Prove: $\forall x \left(\text{Even}(x) \rightarrow \text{Even}(x^2)\right)$

Declare an arbitrary variable for each ∀.	Let x be an arbitrary integer.
Assume the left side of the implication.	Suppose that x is even.
Unroll the predicate definitions.	Then by definition of even, there exists some integer k such that $x = 2k$.
Manipulate towards the goal.	Squaring both sides, we see that: $x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$
Reroll definitions into the right side of the implication.	Because k is an integer, then $2k^2$ is also an integer. So x^2 is two times an integer. So by definition of even, x^2 is even.
Conclude that you have proved the claim.	Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

Direct Proof Template

- Declare an arbitrary variable for each ∀ quantifier
- Assume the left side of the implication
- Unroll the predicate definitions
- Manipulate towards the goal (using creativity, algebra, etc.)
- Reroll definitions into the right side of the implication
- Conclude that you have proved the claim

Another Direct Proof

Prove: "The product of two odd integers is odd."

What's the claim in logic?

How would we prove this claim?

Another Direct Proof

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \Big(\big(\mathrm{Odd}(x) \land \mathrm{Odd}(y)\big) \to \mathrm{Odd}(xy) \Big)$$

A note on Domain of Discourse

"The product of two odd integers is odd."

Domain: Integers

Translation:

Proof Outline:

Domain: Odd Integers

Translation:

Proof Outline:

A note on Translation to Logic

- We first translate the claim to predicate logic because:
  - The translation makes it precise what we are proving
  - The translation hints at the structure of the proof

    e.g. for each ∀, introduce an arbitrary variable
- In practice, computer scientists identify the proof claim and structure without predicate logic translation
- Eventually we'll stop asking you to translate to logic first

Square

Definition:

An integer x is square iff ______________________________________

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k\,(x = k^2)$

Prove: The product of two square integers is square.

What's the claim in logic?

Prove this claim.

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k \left(x = k^2\right)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \Big(\big(\text{Square}(n) \land \text{Square}(m)\big) \rightarrow \text{Square}(nm)\Big)$$

Proof Strategy: Contrapositive

A Direct Proof?

Definitions
$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

What's the claim in logic?

Prove this claim.

Proof by Contrapositive

Proof by contrapositive is another strategy for proving statements of the

form ___________________________________.

The strategy is to ____.

Proof by Contrapositive

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

Another Proof by Contrapositive

Definitions
$\text{Even}(x) := \exists k(x = 2k)$

Prove by Contrapositive: For an integer n, if n^3 is even, then n is even.