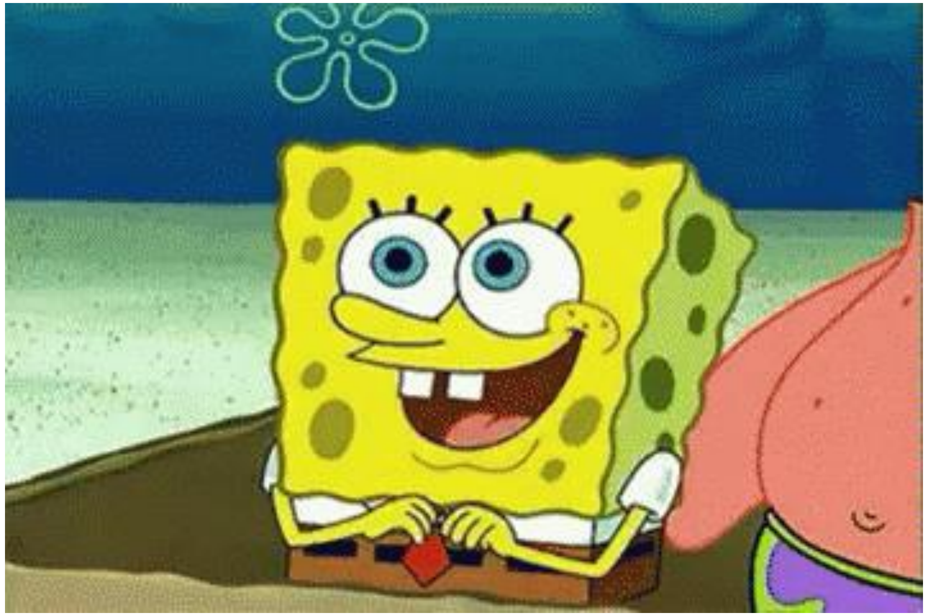

Direct Proof

CSE 311: Foundations of Computing I
Lecture 6

Announcements

- HW1 solutions at the front
- HW2 due Wednesday, 11:59 pm
- Thank you for your feedback!

Review: Predicate Logic

<u>3 Parts</u>

1. Predicate – Function that outputs true or false.
   $\mathrm{Prime}(x) \coloneqq x$ is prime

2. Domain of Discourse – Set of possible inputs to a predicate.
   E.g. Integers

3. Quantifiers – A statement about when a predicate is true
   For all: ∀ There exists: ∃

Theorems and Proofs

Theorems and Proofs

Definition:

A **theorem** is a statement that can be shown to be true.

A **proof** is a valid argument that establishes a statement to be true.

Theorems and Proofs

for all

Examples of theorems include…

- Given a right triangle with side lengths a, b and hypotenuse c, $a^2 + b^2 = c^2$ Pythagorean Theorem
- There are infinitely many prime numbers.
- There exists a problem that cannot be solved by a program.

Halting Problem

there exists

Integer

Definition:

An **integer** is a real number with no fractional part

E.g. −17, 0, 32

Odd and Even

$\underbrace{20}_{x} = 2 \cdot \underbrace{10}_{k}$ $\quad$ $\underbrace{-8}_{x} = 2\underbrace{(-4)}_{k}$

Definitions:

An integer x is even iff there exists some integer k such that.

$x = 2k$

$$\boxed{Even(x) := \exists k (x = 2k)}$$

(domain integers)

An integer x is odd iff there exists some integer k such that.

$$\boxed{odd(x) := \exists k (x = 2k+1)}$$

$x = 2k+1$

$\underbrace{17}_{x} = 2 \cdot \underbrace{8}_{k} + 1$

$\underbrace{-19}_{x} = 2 \cdot \underbrace{(-10)}_{k} + 1$

Proof Strategy: Direct Proof

Direct Proof

Direct proof is one strategy for proving statements of the form

- $\forall x\ (P(x) \rightarrow Q(x))$ __________________________________.

- $\forall y \forall z((P(y) \wedge Q(z)) \rightarrow R(y,z))$

Our First Direct Proof

Prove: "For all integers x, if x is even, then x^2 is even."

What's the claim in logic? (domain: integers)

$$\forall x\,(\,\mathrm{Even}(x) \to \mathrm{Even}(x^2)\,)$$

How would we prove this claim?

<u>scratchwork</u>

suppose x is even

$x = 2k$ for int k

$x^2 = 4k^2 = 2 \cdot (\underline{2k^2})$

x^2 is even

Our First Direct Proof

$x = 10$

$x = 2 \cdot \underbrace{5}_{k}$

$x^2 = (2 \cdot 5)^2 = 4 \cdot 25 = 2 \cdot \underline{50}$

Definitions

$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x \left(\text{Even}(x) \rightarrow \text{Even}(x^2)\right)$

Let x be an arbitrary integer. Suppose x is even.
Then by definition of even, $x = 2k$ for some integer k.
Squaring both sides, we can see that:

$$x^2 = (2k)^2 = 4k^2 = 2\,(\underline{2k^2})$$

Since k is an integer, $2k^2$ is an integer. So x^2 is 2 times an integer. By definition of even, x^2 is even.
Since x was arbitrary, we can conclude that for all integers x, if x is even then x^2 is even. $\square$

Direct Proof Template

Prove: $\forall x\left(\mathrm{Even}(x) \rightarrow \mathrm{Even}(x^2)\right)$

Declare an arbitrary variable for each ∀.	Let x be an arbitrary integer.
Assume the left side of the implication.	Suppose that x is even.
Unroll the predicate definitions.	Then by definition of even, there exists some integer k such that $x = 2k$.
Manipulate towards the goal.	Squaring both sides, we see that: $x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$
Reroll definitions into the right side of the implication.	Because k is an integer, then $2k^2$ is also an integer. So x^2 is two times an integer. So by definition of even, x^2 is even.
Conclude that you have proved the claim.	Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

Direct Proof Template

- Declare an arbitrary variable for each $\forall$ quantifier
- Assume the left side of the implication
- Unroll the predicate definitions
- Manipulate towards the goal (using creativity, algebra, etc.)
- Reroll definitions into the right side of the implication
- Conclude that you have proved the claim

Another Direct Proof

Prove: "The product of two odd integers is odd."

For all odd integers x and odd integers y, xy is odd

What's the claim in logic?

$$\forall x \forall y\ ((\mathrm{odd}(x) \wedge \mathrm{odd}(y)) \rightarrow \mathrm{odd}(xy))$$

How would we prove this claim?

- Let x & y be arbitrary integers.
- Assume x & y are odd
- Show xy is odd

Another Direct Proof

$x = 17 = 2 \cdot \underline{8} + 1$
$y = 5 = 2 \cdot \underline{2} + 1$

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \Big(\big(\mathrm{Odd}(x) \wedge \mathrm{Odd}(y) \big) \rightarrow \mathrm{Odd}(xy) \Big)$$

Let x and y be arbitrary integers. Suppose that x and y are odd. So by def. of odd, $x = 2k+1$ for some integer k. By def of odd, $y = 2j+1$ for some integer j. Then multiplying x and y we can see that:

$xy = (2k+1)(2j+1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1$

Since k and j are integers, $2kj + k + j$ is an integer. Thus xy is 2 times an integer plus 1. So by def. of odd, xy is odd. Since x and y were arbitrary, we have shown that the product of...

A note on Domain of Discourse

"The product of two odd integers is odd."

Domain: Integers

Translation:

$\forall x \forall y ((odd(x) \wedge odd(y)) \to odd(xy))$

Proof Outline:

- Let x, y be arb. integers
- Assume x & y are odd
- Show xy is odd.

Domain: Odd Integers

Translation:

$\forall x \forall y (odd(xy))$

Proof Outline: ☆

- Let x and y be arbitrary odd integers
- Show xy is odd.

A note on Translation to Logic

- We first translate the claim to predicate logic because:
  - The translation makes it precise what we are proving
  - The translation hints at the structure of the proof

    e.g. for each $\forall$, introduce an arbitrary variable
- In practice, computer scientists identify the proof claim and structure without predicate logic translation
- Eventually we'll stop asking you to translate to logic first

Square

$36 = 6^2$

Definition:

An integer x is **square** iff there exists an integer k such that $x = k^2$.

$$\boxed{\text{Square}(x) := \exists k \, (x = k^2)}$$

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k\,(x = k^2)$

Prove: The product of two square integers is square.

$4 \cdot 9 = 36$

What's the claim in logic?

$\forall x\, \forall y\, ((\text{square}(x) \wedge \text{Square}(y)) \rightarrow \text{Square}(xy))$

Prove this claim.

Yet Another Direct Proof

Definitions
$\mathrm{Square}(x) \coloneqq \exists k\,(x = k^2)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \big((\mathrm{Square}(n) \land \mathrm{Square}(m)) \rightarrow \mathrm{Square}(nm) \big)$$

Let n and m be arbitrary integers. Suppose that n and m are square. By definition of square, $n = a^2$ for some integer a and $m = b^2$ for some integer b. Then multiplying n and m, we can see:

$$nm = a^2 \cdot b^2 = (ab)^2$$

Since a and b are integers, ab is an integer. So by def of square, nm is square.

Since n, m were arbitrary, we have shown that the product of two square integers is square.

Proof Strategy: Contrapositive

A Direct Proof?

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

What's the claim in logic?

Prove this claim.

Proof by Contrapositive

Proof by contrapositive is another strategy for proving statements of the

form ______________________________.

The strategy is to ______________________________________.

Proof by Contrapositive

<u>Definitions</u>
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

Another Proof by Contrapositive

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove by Contrapositive: For an integer n, if n^3 is even, then n is even.