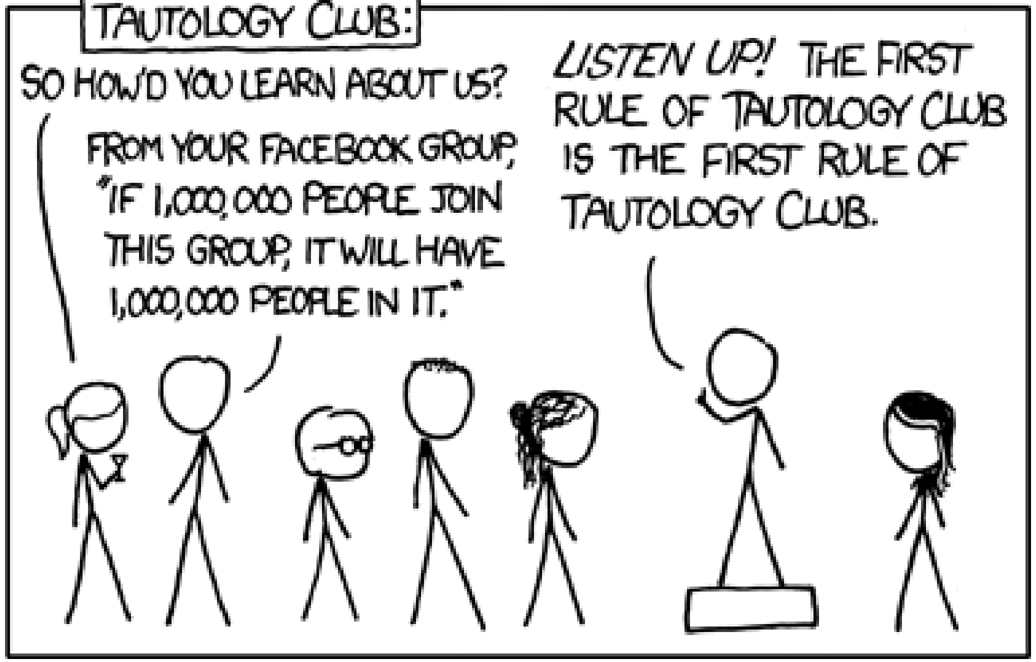


Predicate Logic, Nested Quantifiers

CSE 311: Foundations of Computing I
Lecture 5

Announcements

- HW2
  - Due Wednesday, 11:59 pm
- Lecture on Monday

Review

Predicate Logic

3 Parts

1. Predicate – Function that outputs true or false.
   $\text{Prime}(x) \coloneqq x$ is prime

2. Domain of Discourse – Set of possible inputs to a predicate.
   E.g. Integers

3. Quantifiers – A statement about when a predicate is true
   For all: $\forall$ There exists: $\exists$

Translation

Domain of Discourse
Cats

Predicate Definitions
$\text{Fluffy}(x) \coloneqq x$ is fluffy
$\text{Happy}(x) \coloneqq x$ is happy

If a cat is fluffy, it is happy.

$$\forall x\left(\text{Fluffy}(x) \rightarrow \text{Happy}(x)\right)$$

Translation

Domain of Discourse
Animals

Predicate Definitions
$\text{Cat}(x) := x$ is a cat
$\text{Fluffy}(x) := x$ is fluffy
$\text{Happy}(x) := x$ is happy

If a cat is fluffy, it is happy.

$$\forall x \left(\left(\text{Cat}(x) \wedge \text{Fluffy}(x) \right) \rightarrow \text{Happy}(x) \right)$$

Nested Quantifiers

Example 1

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Humans are not friends with each other.

$$\forall x \forall y \left(\left(\text{Human}(x) \land \text{Human}(y) \right) \rightarrow \neg \text{Friends}(x, y) \right)$$

Example 2

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

All humans are friends with the dogs that they walk.

$$\forall x \forall y \left(\left(\text{Human}(x) \wedge \text{Dog}(y) \wedge \text{Walks}(x, y) \right) \rightarrow \text{Friends}(x, y) \right)$$

Example 3

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Every human walks a dog.

Example 3

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Every human walks a dog.

a) $\forall x \exists y \big(\text{Human}(x) \wedge \text{Dog}(y) \wedge \text{Walks}(x, y)\big)$

b) $\forall x \exists y \Big(\big(\text{Human}(x) \wedge \text{Dog}(y)\big) \rightarrow \text{Walks}(x, y)\Big)$

c) $\forall \boldsymbol{x} \exists \boldsymbol{y} \Big(\textbf{Human}(\boldsymbol{x}) \rightarrow \big(\textbf{Dog}(\boldsymbol{y}) \wedge \textbf{Walks}(\boldsymbol{x}, \boldsymbol{y})\big)\Big)$

d) $\forall x \exists y \Big(\text{Human}(x) \wedge \big(\text{Dog}(y) \rightarrow \text{Walks}(x, y)\big)\Big)$

Poll Everywhere
pollev.com/anjalia

Example 4

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Every human walks exactly one dog.

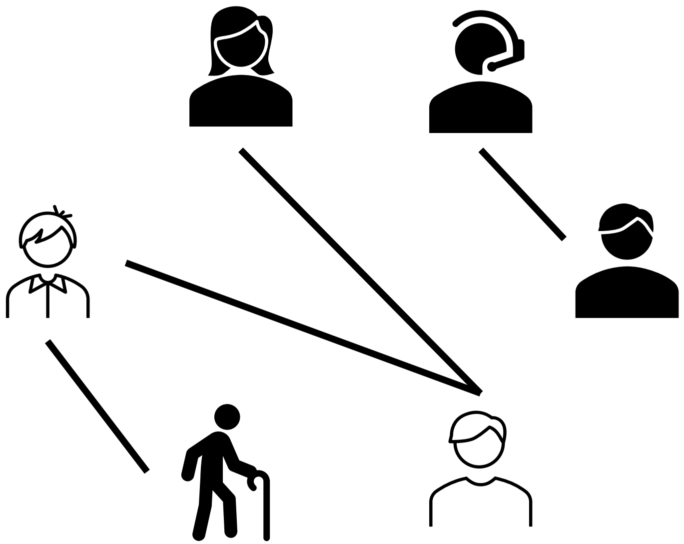
$$\forall x[\text{Human}(x) \rightarrow \exists y(\text{Dog}(y) \wedge \text{Walks}(x, y) \wedge \forall z(\text{Dog}(z) \wedge (z \neq y) \rightarrow \neg\text{Walks}(x, z)))]$$

Quantifier Order

Quantifier Order

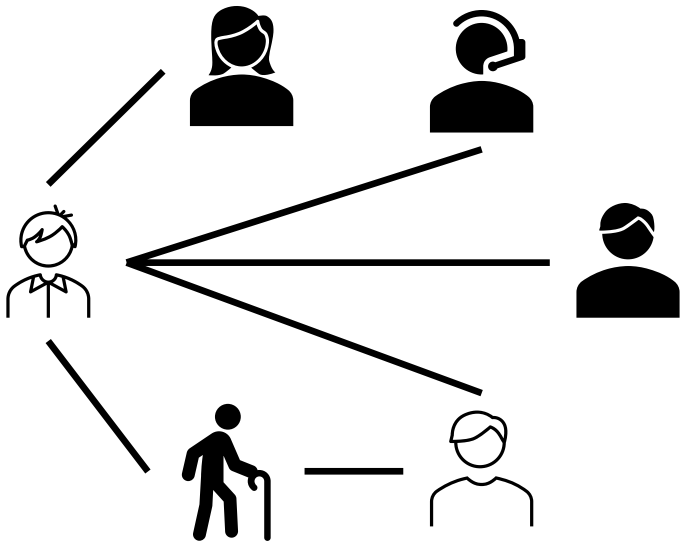
Translate to logic. The domain of discourse is people. The predicate $\texttt{Friends}(x, y)$ is defined as x and y are friends.

Everyone is friends with someone.



$\forall x \exists y \; \texttt{Friends}(x, y)$

Someone is friends with everyone.



$\exists y \forall x \; \texttt{Friends}(x, y)$

Quantifier Order

$\forall x \exists y\, \mathrm{P}(x, y)$ means for every x, there exists a y such that $\mathrm{P}(x, y)$ is true.

$\exists y \forall x\, \mathrm{P}(x, y)$ means there exists some y such that for all x, $\mathrm{P}(x, y)$ is true.

Quantifier Order

$\forall x \exists y \text{ Likes}(x, y)$

Everyone has some person they like.

$\forall y \exists x \text{ Likes}(x, y)$

Everyone has some person that likes them.

$\exists x \forall y \text{ Likes}(x, y)$

There is a person that likes everyone.

$\exists y \forall x \text{ Likes}(x, y)$

There is a person that everyone likes.

Quantifier Order

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y\, P(x, y)$

$\forall x \exists y\, P(x, y)$

y

x

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Quantifier Order

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $\mathrm{P}(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y\ \mathrm{P}(x, y)$

A row, where every entry is T

$\forall x \exists y\ \mathrm{P}(x, y)$

In every row there must be a T

y (columns), x (rows)

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Negating Quantifiers

DeMorgan's Law for Quantifiers

Consider the following sentences:

- There does not exist a green penguin.
- Every penguin is a color other than green.

Are they logically equivalent?

DeMorgan's Law for Quantifiers

Consider the following sentences:

- Not every person can dance.
- There is a person that cannot dance.

Are they logically equivalent?

DeMorgan's Law for Quantifiers

$$\boxed{\begin{array}{c} \neg\forall x\ \mathrm{P}(x) \equiv \exists x\ \neg\mathrm{P}(x) \\ \neg\exists x\ \mathrm{P}(x) \equiv \forall x\ \neg\mathrm{P}(x) \end{array}}$$

I.e. to negate an expression with a quantifier:

1. Switch the quantifier (∀ becomes ∃, and vice versa).
2. Negate the expression inside.

Example 1

Translate to predicate logic & rewrite using DeMorgan’s Law.

There is no integer which is prime and even.

$$\neg\exists x\big(\mathrm{Prime}(x) \wedge \mathrm{Even}(x)\big)$$

$$\equiv \forall x\neg\big(\mathrm{Prime}(x) \wedge \mathrm{Even}(x)\big)$$

$$\equiv \forall x\big(\neg\mathrm{Prime}(x) \vee \neg\mathrm{Even}(x)\big)$$

All integers are not prime or not even.

Example 2

Translate to predicate logic & rewrite using DeMorgan's Law.

There is no integer greater than or equal to every other integer.

$$\neg\exists x\, \forall y\, (x \geq y)$$

$$\equiv \forall x\, \neg\, \forall y\, (x \geq y)$$

$$\equiv \forall x\, \exists y\, \neg(x \geq y)$$

$$\equiv \forall x\, \exists y\, (x < y)$$

For every integer, there is an integer greater than it.

Predicate Logic Equivalence

Motivation

- We saw with the last two examples that there may be different predicate logic expressions that have the same meaning
- We can prove logical equivalence of Predicate Logic statements like we did for Propositional Logic
- Same equivalence rules still apply, in addition to DeMorgan’s Law for Quantifiers

Proving Predicate Logic Equivalence

"No odd integer is equal to an even integer."

Alice translated this as: $\neg\exists x \exists y \left(\text{Odd}(x) \wedge \text{Even}(y) \wedge (x = y)\right)$

Bob translated this as: $\forall x \forall y\left(\text{Odd}(x) \wedge \text{Even}(y) \rightarrow (x \neq y)\right)$

Prove that these translations are logically equivalent.

Proving Predicate Logic Equivalence

$\neg\exists x\exists y\left(\mathrm{Odd}(x)\wedge\mathrm{Even}(y)\wedge(x=y)\right)$

$\equiv\forall x\neg\exists y\left(\mathrm{Odd}(x)\wedge\mathrm{Even}(y)\wedge(x=y)\right)$ DeMorgan's Law for Quantifiers

$\equiv\forall x\,\forall y\,\neg\left(\mathrm{Odd}(x)\wedge\mathrm{Even}(y)\wedge(x=y)\right)$ DeMorgan's Law for Quantifiers

$\equiv\forall x\,\forall y\,\left(\neg\left(\mathrm{Odd}(x)\wedge\mathrm{Even}(y)\right)\vee\neg(x=y)\right)$ DeMorgan's Law

$\equiv\forall x\forall y\left(\neg\left(\mathrm{Odd}(x)\wedge\mathrm{Even}(y)\right)\vee(x\neq y)\right)$ Definition of $\neq$

$\equiv\forall x\forall y\left(\left(\mathrm{Odd}(x)\wedge\mathrm{Even}(y)\right)\rightarrow(x\neq y)\right)$ Law of Implication

Proving Predicate Logic Equivalence

$\neg\forall x\left(\mathrm{P}(x) \to \exists y\, \mathrm{Q}(x,y)\right)$

$\equiv \neg\forall x\left(\neg\mathrm{P}(x) \vee \exists y\, \mathrm{Q}(x,y)\right)$ Law of Implication

$\equiv \exists x\, \neg\left(\neg\mathrm{P}(x) \vee \exists y\, \mathrm{Q}(x,y)\right)$ DeMorgan's Law for Quantifiers

$\equiv \exists x\left(\mathrm{P}(x) \wedge \neg\exists y\, \mathrm{Q}(x,y)\right)$ DeMorgan's Law

$\equiv \exists x\left(\mathrm{P}(x) \wedge \forall y\, \neg\mathrm{Q}(x,y)\right)$ DeMorgan's Law for Quantifiers

$\equiv \exists x \forall y\left(\mathrm{P}(x) \wedge \neg\mathrm{Q}(x,y)\right)$

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