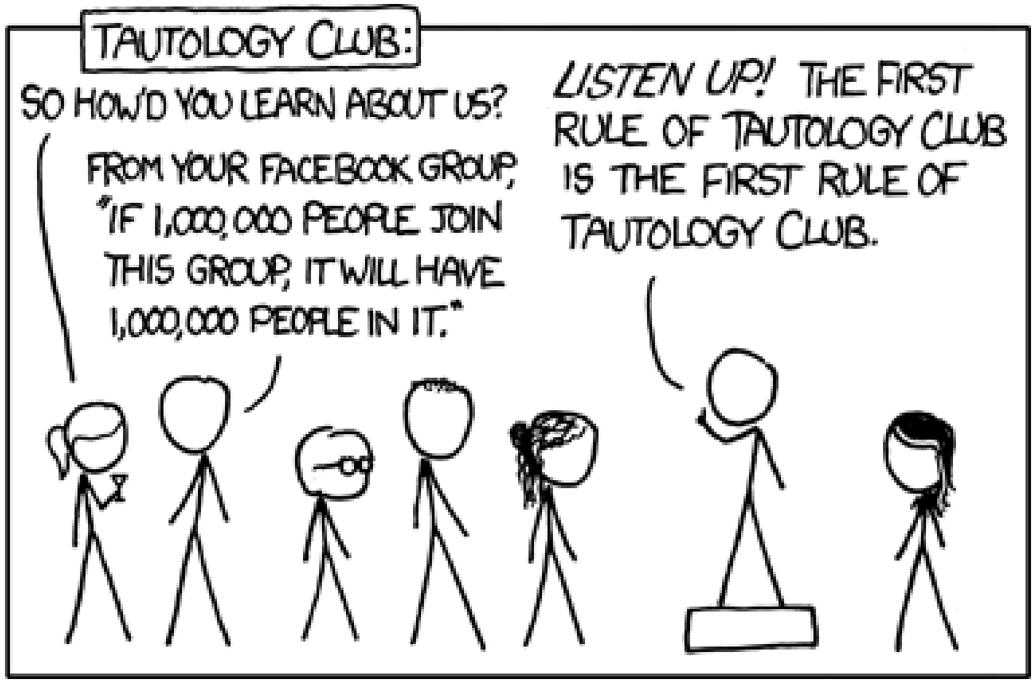


Predicate Logic, Nested Quantifiers

CSE 311: Foundations of Computing I
Lecture 5

Announcements

- HW2
  - Due Wednesday, 11:59 pm
- Lecture on Monday

Review

Predicate Logic

<u>3 Parts</u>

1. Predicate – Function that outputs true or false.
   $\text{Prime}(x) \coloneqq x$ is prime

2. Domain of Discourse – Set of possible inputs to a predicate.
   E.g. Integers

3. Quantifiers – A statement about when a predicate is true
   For all: $\forall$ There exists: $\exists$

Translation

Domain of Discourse
Cats

Predicate Definitions
$\text{Fluffy}(x) \coloneqq x$ is fluffy
$\text{Happy}(x) \coloneqq x$ is happy

If a cat is fluffy, it is happy.

For all cats x, if x is fluffy, then x is happy

$\forall x\,(\text{fluffy}(x) \rightarrow \text{Happy}(x))$

Translation

Domain of Discourse
Animals

← domain restriction

Predicate Definitions
$\text{Cat}(x) :=$ x is a cat
$\text{Fluffy}(x) :=$ x is fluffy
$\text{Happy}(x) :=$ x is happy

If a cat is fluffy, it is happy.

✓ $\forall x\,((\text{Cat}(x) \wedge \text{Fluffy}(x)) \to \text{Happy}(x))$

✗ $\forall x\,(\text{Cat}(x) \wedge (\text{Fluffy}(x) \to \text{Happy}(x)))$

Nested Quantifiers

Example 1

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Humans are not friends with each other.

For all humans x, they are not friends with all humans y.

$$\forall x \forall y \, ((\text{Human}(x) \wedge \text{Human}(y)) \rightarrow \neg \text{Friends}(x, y))$$

```
for (Human x : Humans):
    for (Human y: Humans):
        assert !Friends(x, y)
```

Example 2

Domain of Discourse
Mammals

Predicate Definitions
- $\text{Walks}(x, y) \coloneqq x$ walks y
- $\text{Friends}(x, y) \coloneqq x$ and y are friends
- $\text{Human}(x) \coloneqq x$ is a human
- $\text{Dog}(x) \coloneqq x$ is a dog

$\forall x$ $\forall y$

All humans are friends with the dogs that they walk.

$$\forall x \forall y ((\text{Human}(x) \wedge \text{Dog}(y) \wedge \text{Walks}(x,y)) \to \text{Friends}(x,y))$$

Example 3

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Every human walks a dog.

$\forall$ $\exists$

$$\forall x \left([\text{Human}(x)] \rightarrow [\exists y\, (\text{Dog}(y) \wedge \text{Walks}(x,y))] \right)$$

$$\equiv \forall x \exists y\, (\text{Human}(x) \rightarrow (\text{Dog}(y) \wedge \text{Walks}(x,y)))$$

Example 3

Domain of Discourse
Mammals

Predicate Definitions
$\mathrm{Walks}(x, y) \coloneqq x$ walks y
$\mathrm{Friends}(x, y) \coloneqq x$ and y are friends
$\mathrm{Human}(x) \coloneqq x$ is a human
$\mathrm{Dog}(x) \coloneqq x$ is a dog

Every human walks a dog.

~~a)~~ $\forall x \exists y \big(\mathrm{Human}(x) \land \mathrm{Dog}(y) \land \mathrm{Walks}(x, y)\big)$

~~b)~~ $\forall x \exists y \Big(\big(\mathrm{Human}(x) \land \mathrm{Dog}(y)\big) \to \mathrm{Walks}(x, y)\Big)$

y: Rabbit

c) $\forall x \exists y \Big(\mathrm{Human}(x) \to \big(\mathrm{Dog}(y) \land \mathrm{Walks}(x, y)\big)\Big)$

~~d)~~ $\forall x \exists y \Big(\mathrm{Human}(x) \land \big(\mathrm{Dog}(y) \to \mathrm{Walks}(x, y)\big)\Big)$

Poll Everywhere
pollev.com/anjalia

Example 4

Domain of Discourse
Mammals

Predicate Definitions
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Every human walks exactly one dog.

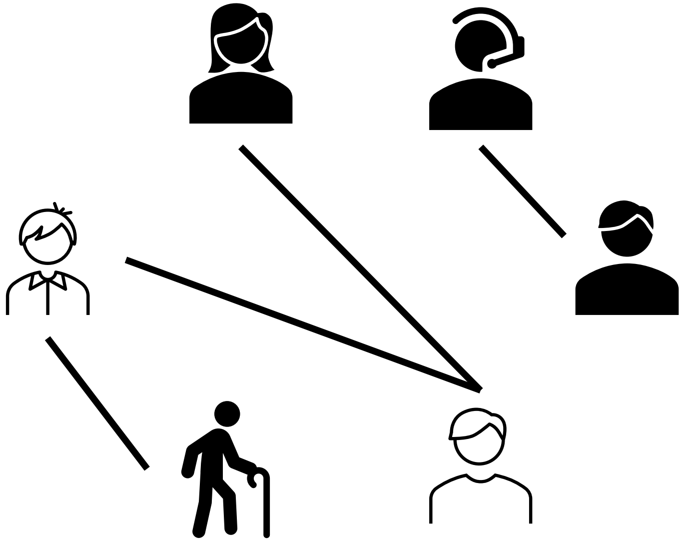
$$\forall x\,(\text{Human}(x) \rightarrow \exists y\,(\text{Dog}(y) \wedge \text{Walks}(x,y) \wedge$$
$$\forall z\,((\text{Dog}(z) \wedge (z \neq y)) \rightarrow \neg \text{Walks}(x,z)))$$
$$\forall z\,((\text{Dog}(z) \wedge \text{Walks}(x,z)) \rightarrow (z=y)))$$

Quantifier Order

Quantifier Order

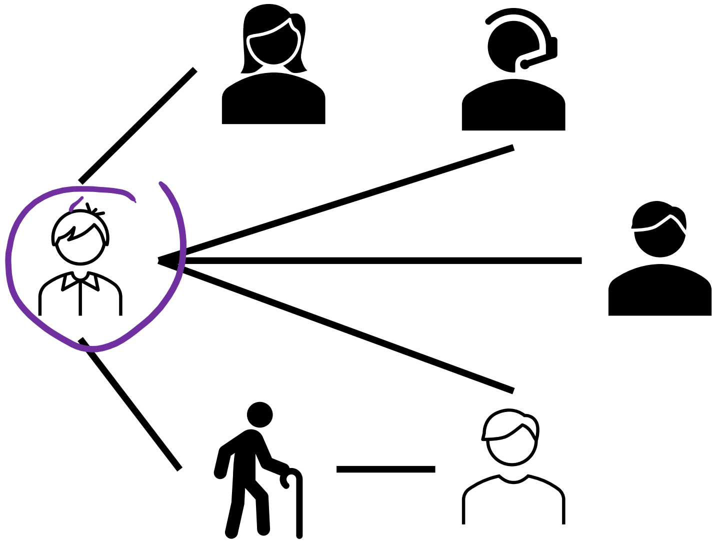
Translate to logic. The domain of discourse is people. The predicate `Friends`(x, y) is defined as x and y are friends.

Everyone is friends with someone.



$\forall x \exists y\, (\text{Friends}(x, y))$

Someone is friends with everyone.



$\exists y \forall x\, (\text{Friends}(x, y))$

Quantifier Order

$\forall x \exists y$ P(x, y) means for every x, there is a y that makes P(x, y) true (y could vary).

$\exists y \forall x$ P(x, y) means there is some special y that makes P(x, y) true for every x.

Quantifier Order

Domain: people
Likes(x, y) := x likes y

$\forall x \exists y \, Likes(x, y)$

Everyone likes someone.

$\forall y \exists x \, Likes(x, y)$

Everyone is liked by someone.

$\exists x \forall y \, Likes(x, y)$

There is someone who likes everyone.

$\exists y \forall x \, Likes(x, y)$

There is someone who everyone likes.

Quantifier Order

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y\, P(x, y)$

$\forall x \exists y\, P(x, y)$

y

x

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Quantifier Order

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y\ P(x, y)$

A row, where every entry is T

$\forall x \exists y\ P(x, y)$

In every row there must be a T

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Negating Quantifiers

DeMorgan's Law for Quantifiers

Domain: penguins
Predicate: Green

Consider the following sentences:

- There does not exist a green penguin. $\neg \exists x\ (Green(x))$
- Every penguin is a color other than green. $\forall x\ (\neg Green(x))$

Are they logically equivalent? ... Yes?

DeMorgan's Law for Quantifiers

Consider the following sentences:

- Not every person can dance. $\neg \forall x\, (\mathrm{Dance}(x))$
- There is a person that cannot dance. $\exists x\, (\neg \mathrm{Dance}(x))$

Are they logically equivalent? -- yes.

DeMorgan’s Law for Quantifiers

$$\boxed{\begin{aligned} \neg\forall x\, \mathrm{P}(x) &\equiv \exists x\, \neg\mathrm{P}(x) \\ \neg\exists x\, \mathrm{P}(x) &\equiv \forall x\, \neg\mathrm{P}(x) \end{aligned}}$$

I.e. to negate an expression with a quantifier:

1. Switch the quantifier ($\forall$ becomes $\exists$, and vice versa).
2. Negate the expression inside.

Example 1

Translate to predicate logic & rewrite using DeMorgan's Law.

There is no integer which is prime and even.

$\neg \exists x \, (Prime(x) \wedge Even(x))$

$\equiv \forall x \neg (Prime(x) \wedge Even(x))$

$\equiv \forall x \, (\neg Prime(x) \vee \neg Even(x))$

All integers are not prime or not even.

Example 2

Translate to predicate logic & rewrite using DeMorgan's Law.

There is no integer greater than or equal to every other integer.

$$\neg \exists x \, \forall y \, (x \geq y)$$

$$\equiv \forall x \, \neg \forall y \, (x \geq y)$$

$$\equiv \forall x \, \exists y \, \neg (x \geq y)$$

$$\equiv \forall x \, \exists y \, (x < y)$$

All integers have an integer greater than them.

Predicate Logic Equivalence

Motivation

- We saw with the last two examples that there may be different predicate logic expressions that have the same meaning
- We can prove logical equivalence of Predicate Logic statements like we did for Propositional Logic
- Same equivalence rules still apply, in addition to DeMorgan’s Law for Quantifiers

Proving Predicate Logic Equivalence

"No odd integer is equal to an even integer."

Alice translated this as: $\neg \exists x \exists y \, (Odd(x) \wedge Even(y) \wedge (x = y))$

Bob translated this as: $\forall x \forall y \, ((Odd(x) \wedge Even(y)) \rightarrow (x \neq y))$

Prove that these translations are logically equivalent.

Proving Predicate Logic Equivalence

$\neg\exists x \exists y \left(\text{Odd}(x) \wedge \text{Even}(y) \wedge (x = y)\right)$

$\equiv \forall x \forall y \neg((\text{Odd}(x) \wedge \text{Even}(y)) \wedge (x = y))$ DeMorgan's (x2)

$\equiv \forall x \forall y (\neg(\text{Odd}(x) \wedge \text{Even}(y)) \vee \neg(x = y))$ DeMorgan's

$\equiv \forall x \forall y (\neg(\text{Odd}(x) \wedge \text{Even}(y)) \vee (x \neq y))$ Def of $\neq$

$\equiv \forall x \forall y \left(\left(\text{Odd}(x) \wedge \text{Even}(y)\right) \rightarrow (x \neq y)\right)$ Law of Imp.

Proving Predicate Logic Equivalence

$\neg\forall x\left(P(x) \to \exists y\, Q(x,y)\right)$

$\equiv \neg\forall x\,\exists y\,(P(x) \to Q(x,y))$

$\equiv \exists x\,\forall y\,\neg(P(x) \to Q(x,y))$ De Morgan's (x2)

$\equiv \exists x\,\forall y\,\neg(\neg P(x) \vee Q(x,y))$ Law of imp.

$\equiv \exists x\,\forall y\,(\neg\neg P(x) \wedge \neg Q(x,y))$ De Morgan's Law

$\equiv \exists x \forall y\left(P(x) \wedge \neg Q(x,y)\right)$ Double Negation

Anonymous Feedback

https://tinyurl.com/cse311feedback