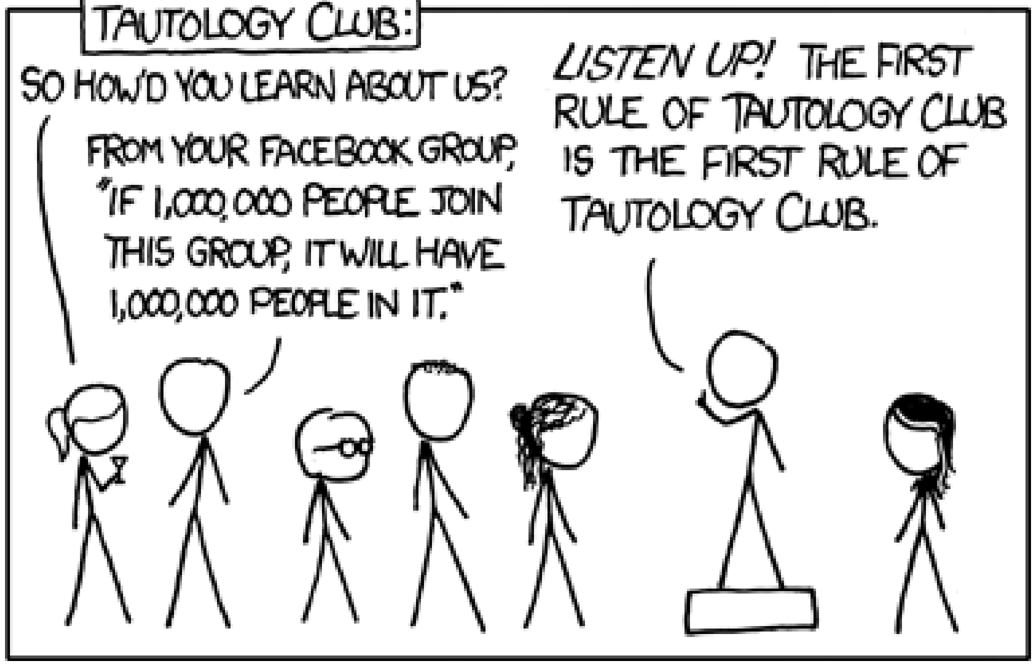
Warm Up

p	q	r	$F(p,q,r)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

1. Write a propositional logic expression for F in DNF (ORs of ANDs) form

2. Write a propositional logic expression for F in CNF (ANDs of ORs) form



Predicate Logic

CSE 311: Foundations of Computing I
Lecture 4

Announcements

- HW1
  - Due tonight, 11:59 pm
- HW2
  - Released tonight

Warm Up

p	q	r	$F(p,q,r)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

1. Write a propositional logic expression for F in DNF (ORs of ANDs) form
   $(p \land q \land r) \lor (p \land q \land \neg r) \lor (p \land \neg q \land \neg r) \lor (\neg p \land q \land r) \lor (\neg p \land \neg q \land \neg r)$

2. Write a propositional logic expression for F in CNF (ANDs of ORs) form
   $(\neg p \lor q \lor \neg r) \land (p \lor \neg q \lor r) \land (p \lor q \lor \neg r)$

Boolean Algebra & Circuits

Notation

- Logic is fundamental
  - Computer scientists use it in programs
  - Mathematicians use it in proofs
  - Engineers use it in hardware
  - Philosophers use it in arguments
- Consequently, everyone has their own notation

Boolean Algebra

Another notation for logic. Preferred by some because it's compact.

Term	Propositional Logic	Boolean Algebra
or	$\vee$	$+$
and	$\wedge$	$\cdot$
not	$\neg$	$'$
True	$\mathbf{T}$	$\mathbf{1}$
False	$\mathbf{F}$	$\mathbf{0}$

$(p \wedge q \wedge r) \vee s \vee \neg t$ in Boolean Algebra is $pqr + s + t'$.

Digital Circuits

Computing with Logic

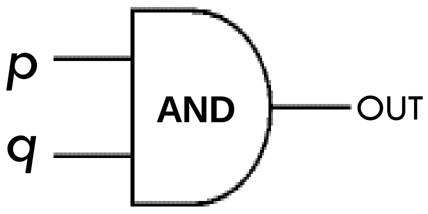
- T corresponds to 1, or high voltage
- F corresponds to 0, or low voltage

Gates

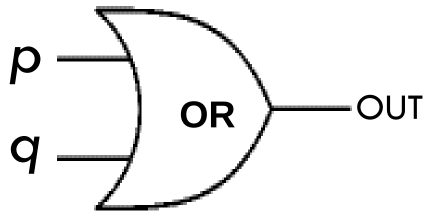
- Gates take inputs and produce outputs

Digital Circuits – Gates

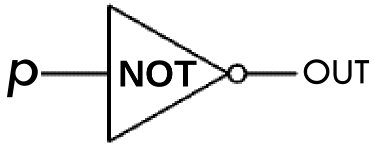
AND Gate



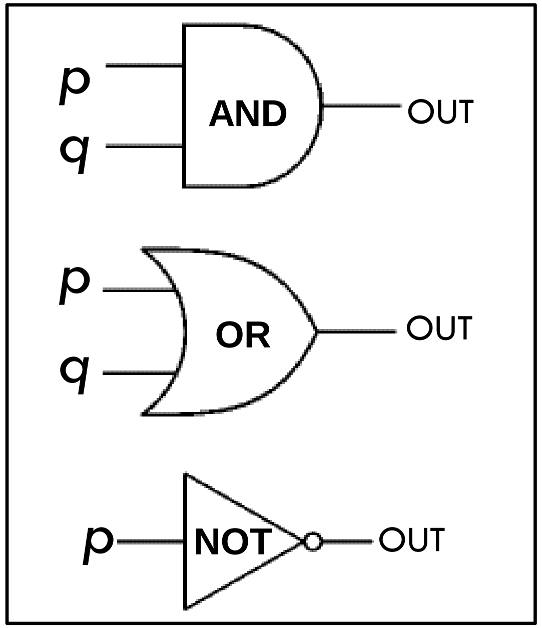
OR Gate



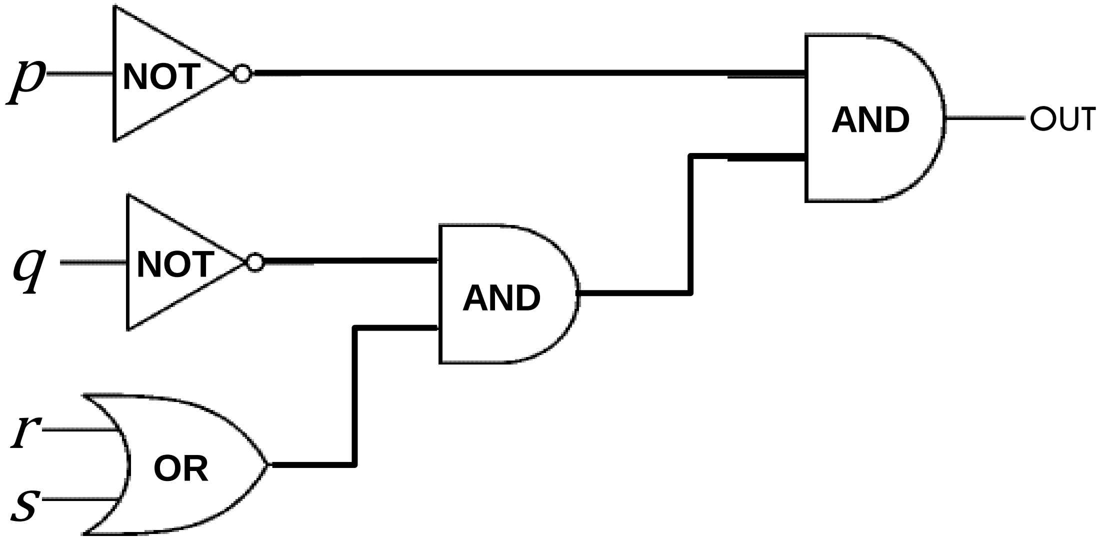
NOT Gate



Digital Circuits – Example



Write $\neg p \wedge (\neg q \wedge (r \vee s))$ as a circuit.



Predicate Logic

Motivation

Often we will work with statements of the form:

If $x > 10$, then $x^2 > 100$.

If x is even, x^2 is even.

Can you translate these to propositional logic?

No. We need a function that is true or false depending on the value of x.

Motivation

Propositional Logic

Lets us break down complex true or false statements into atomic parts joined by connectives.

Predicate Logic

Lets us analyze complex true or false statements that are functions of some underlying objects.

Predicate Logic

3 Parts

1. Predicate
2. Domain of Discourse
3. Quantifiers

Predicate

Definition:

A predicate is a function that outputs true or false.

$\text{Cat}(x) \coloneqq x$ is a cat

$\text{Even}(x) \coloneqq x$ is even

$\text{LessThan}(x, y) \coloneqq x < y$

$\text{Sum}(x, y, z) \coloneqq x + y = z$

$\text{IsRaining} \coloneqq$ It is raining outside

Analogy

Propositions were like Boolean variables.

```
boolean itIsRaining = true
```

Predicates are like functions that return Boolean values.

```
public boolean Even(int x) {…}
```

Predicate Translation: Example

x is prime or x^2 is odd, or $x = 2$

$\text{Prime}(a) \coloneqq a$ is Prime

$\text{Odd}(a) \coloneqq a$ is Odd

$\text{Equals}(a, b) \coloneqq \; a = b$

$$\text{Prime}(x) \vee \text{Odd}(x^2) \vee \text{Equals}(x, 2)$$

Domain of Discourse

Definition:

The **domain of discourse** for a predicate is the set of possible inputs.

$\text{Cat}(x)$ – Possible domains include mammals, animals, cats.

$\text{LessThan}(x, y)$ – Possible domains include numbers, integers.

Domain of Discourse: Example

What's a possible domain of discourse for these predicates?

1. $\text{Prime}(x) \coloneqq x$ is prime

   Positive Integers, Integers, all Numbers

2. $\text{Equals}(x, y) \coloneqq x$ and y are the same object

   Integers, all Numbers, all People, all Mammals

3. $\text{EnrolledIn}(x, y) \coloneqq x$ is enrolled in course y

   Students and Courses

Quantifiers: Motivation

We tend to use variables for two reasons:

1. The statement is true for every x.
   For every integer x, if x is even then x^2 is even.

2. There is some x for which the statement is true.
   There is some problem x that computers cannot solve.

Quantifiers

$\forall x\ \mathrm{P}(x)$

- $\forall$ is called the Universal Quantifier.
- Read out loud as “for all x, P of x”.
- $\forall x\ \mathrm{P}(x)$ means for every x in the domain, $\mathrm{P}(x)$ is true.

$\exists x\ \mathrm{P}(x)$

- $\exists$ is called the Existensial Quantifier.
- Read out loud as “there exists x, P of x”.
- $\exists x\ \mathrm{P}(x)$ means there is some x in the domain for which $\mathrm{P}(x)$ is true.

Predicate Logic Summary

<u>3 Parts</u>

1. Predicate – Function that outputs true or false.

2. Domain of Discourse – Set of possible inputs to a predicate.

3. Quantifiers – A statement about when a predicate is true: ∀ or ∃

Predicate Logic Translation

English to Logic

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \coloneqq x$ is even
$\text{Equals}(x, y) \coloneqq x = y$
$\text{LessThan}(x, y) \coloneqq x < y$

Translate to predicate logic. Then evaluate if the statement is $\mathbf{T}$ or $\mathbf{F}$.

For every integer x, if x is even, then $x = 2$.

$\forall x\big(\text{Even}(x) \rightarrow \text{Equals}(x, 2)\big)$

False, e.g. $x = 4$

There are integers x and y such that $x < y$.

$\exists x \exists y\big(\text{LessThan}(x, y)\big)$

True, e.g. $x = 2, y = 3$

Logic to English

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \coloneqq x$ is even
$\text{LessThan}(x, y) \coloneqq x < y$
$\text{Odd}(x) \coloneqq x$ is odd

Translate to English. Then evaluate if the statement is $\mathbf{T}$ or $\mathbf{F}$.

$\exists x(\text{Odd}(x) \land \text{LessThan}(x, 5))$

There is an odd integer less than 5.

True, e.g. $x = 3$.

$\forall y(\text{Even}(y) \land \text{Odd}(y))$

All integers are even and odd.

False.

Examples

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \coloneqq x$ is even $\text{Greater}(x, y) \coloneqq x > y$
$\text{Odd}(x) \coloneqq x$ is odd $\text{Prime}(x) \coloneqq x$ is prime

Translate to English. Then evaluate if the statement is T or F.

$\exists x\ \text{Even}(x)$	There is an even integer.	T
$\forall x\ \text{Odd}(x)$	All integers are odd.	F
$\forall x\ \big(\text{Even}(x) \vee \text{Odd}(x)\big)$	All integers are even or odd.	T
$\exists x\ \big(\text{Even}(x) \wedge \text{Odd}(x)\big)$	There's an integer that's even and odd.	F
$\forall x\ \text{Greater}(x + 1, x)$	For all integers x, $x + 1 > x$	T
$\exists x\big(\text{Even}(x) \wedge \text{Prime}(x)\big)$	There is an integer that's even and prime.	T

Domain Restriction

Domain Restriction

Definition:

Domain restriction is the technique of limiting our domain of discourse to a smaller set of objects.

Domain Restriction

Domain of Discourse
Animals

Predicate Definitions
$\text{Cat}(x) \coloneqq x$ is a cat
$\text{Blue}(x) \coloneqq x$ is blue

All cats are blue.

$\forall x(\text{Cat}(x) \rightarrow \text{Blue}(x))$

There is a blue cat.

$\exists x(\text{Cat}(x) \land \text{Blue}(x))$

Domain Restriction

Domain of Discourse
Animals

Predicate Definitions
$\text{Cat}(x) \coloneqq x$ is a cat
$\text{Blue}(x) \coloneqq x$ is blue

$\forall x(\text{Cat}(x) \to \text{Blue}(x))$ vs. $\forall x(\text{Cat}(x) \land \text{Blue}(x))$

All cats are blue.

All animals are blue cats.

$\exists x(\text{Cat}(x) \to \text{Blue}(x))$ vs. $\exists x(\text{Cat}(x) \land \text{Blue}(x))$

There is an animal that is not a cat or is blue

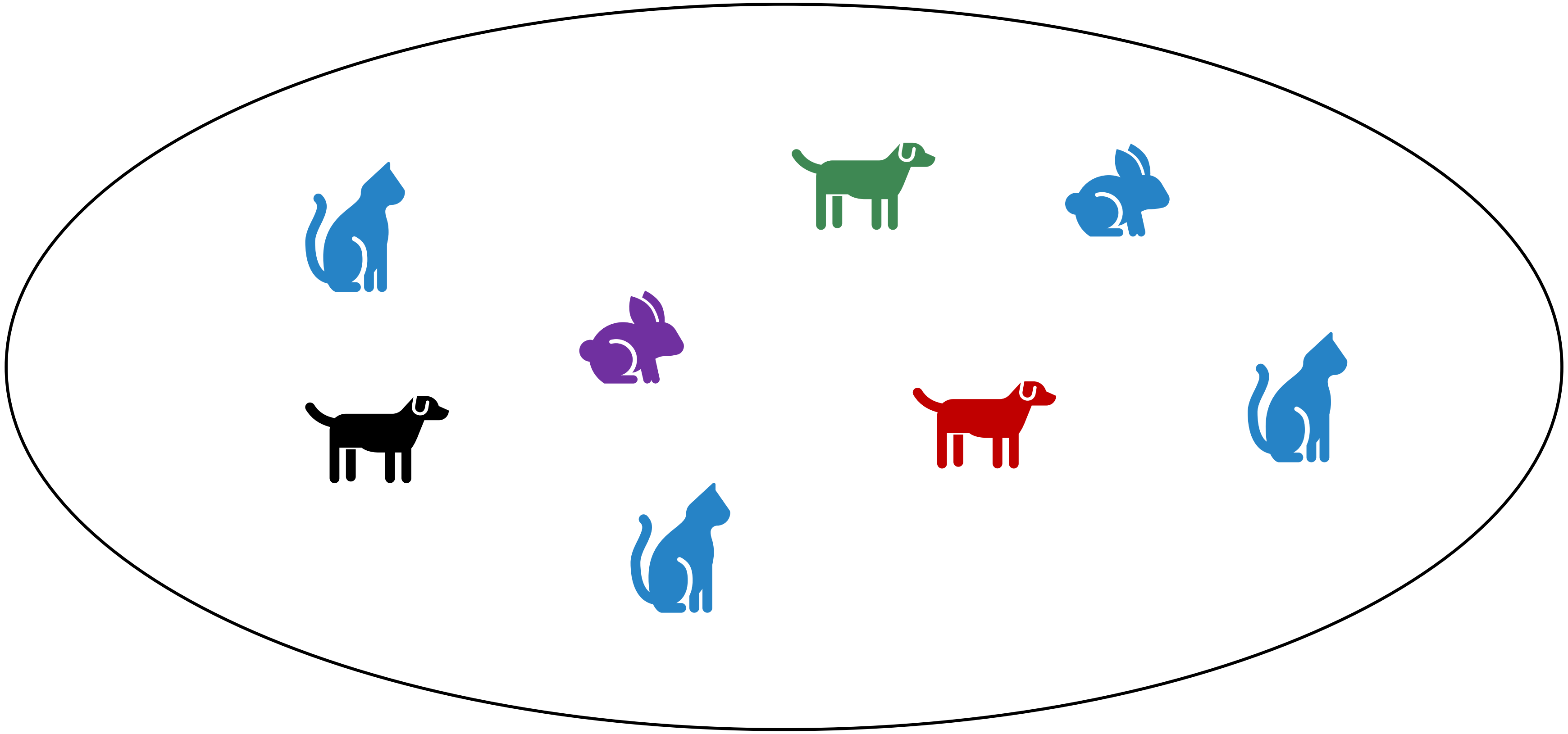
There is a blue cat.

Tip: Avoid $\to$ under $\exists$ in most situations

∀ and vacuous truth

$\forall x(\mathrm{Cat}(x) \rightarrow \mathrm{Blue}(x))$ means "all cats are blue".

One way to check if $\forall x\ \mathrm{P}(x)$ is true is to "loop" over every element of the domain and check that $\mathrm{P}(x)$ is true.



Domain Restriction

Translations often sound more **natural** if we:

1. Notice domain restriction patterns.
2. Avoid using variables when we can.
3. Drop the "for all" or "there exists" when we can.

For example:

$$\forall x(\mathrm{Cat}(x) \rightarrow \mathrm{Blue}(x))$$

✗ For all animals x, if x is a cat then x is blue.

✓ All cats are blue.

Domain Restriction

Domain of Discourse
Food

Predicate Definitions
$\text{Fruit}(x) \coloneqq x$ is a fruit
$\text{Tasty}(x) \coloneqq x$ is tasty
$\text{Ripe}(x) \coloneqq x$ is ripe

Translate these sentences using a **natural-sounding** translation.

$\exists x\left(\text{Fruit}(x) \wedge \text{Tasty}(x)\right)$

There is a tasty fruit. OR Some fruits are tasty.

$\forall x\left(\left(\text{Fruit}(x) \wedge \neg\text{Ripe}(x)\right) \rightarrow \neg\text{Tasty}(x)\right)$

All fruits that aren't ripe aren't tasty.

Quantifier Scope

$\exists x \left(P(x) \land Q(x)\right)$ vs. $\exists x\ P(x) \land \exists x\ Q(x)$

Could be different x's

For example

Domain of Discourse: Integers

$P(x) \coloneqq x$ is odd

$Q(x) \coloneqq x$ is even

Section Tomorrow

Practice with equivalences & translating

Quantifiers can be nested!

There is a person that all people love.

$\exists x \forall y (\text{Love}(y, x))$