

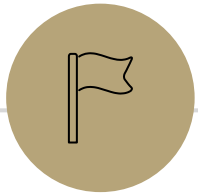


# Equivalences, Normal Forms

CSE 311: Foundations of  
Computing I  
Lecture 3

# Announcements

- HW1
  - Due Wednesday, 11:59 pm
  - Submit on Gradescope
- Office Hours begin today



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## Review

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# Propositions

## Propositions

- Propositions are T/F-valued variables
- Combine using logical connectives ( $\neg, \vee, \wedge, \rightarrow$ , etc.)
- Can be described by a truth table

## Applications

- Understanding complex English sentences
- Simplifying Boolean logic in code

# Logical Equivalence

$A \equiv B$  is an assertion that two propositions  $A, B$  always have the same truth values.

There are two generic methods to proving equivalence:

1. Make a truth table for both propositions
2. Use a chain of logical equivalences

# Recall: Equivalence Laws

Double Negation

$$\neg\neg p \equiv p$$

DeMorgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Commutativity

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

Associativity

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

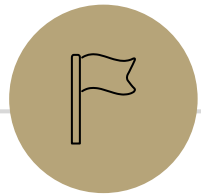
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

# Exercise

Prove that  $p \rightarrow q \equiv \neg q \rightarrow \neg p$  using the logical equivalences rules we've discussed so far.

Do not use contrapositive in the proof.

$$\begin{aligned}\neg q \rightarrow \neg p &\equiv \neg \neg q \vee \neg p && \text{Law of Implication} \\ &\equiv q \vee \neg p && \text{Double Negation} \\ &\equiv \neg p \vee q && \text{Commutativity} \\ &\equiv p \rightarrow q && \text{Law of Implication}\end{aligned}$$



## Logical Equivalences Cont.

# Distributivity

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

# Distributivity: Intuition

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

You go to class, and you read the notes or the textbook.

You go to class and read the notes, or you go to class and you read the textbook.

# Identity

$$p \wedge T \equiv p$$

$$p \vee F \equiv p$$

$p$	$p \wedge T$	$p \vee F$
T	T	T
F	F	F

# Domination

$$p \vee T \equiv T$$

$$p \wedge F \equiv F$$

$p$	$p \vee T$	$p \wedge F$
T	T	F
F	T	F

# Idempotency

$$p \vee p \equiv p$$

$$p \wedge p \equiv p$$

$p$	$p \vee p$	$p \wedge p$
T	T	T
F	F	F

# Negation

$$p \vee \neg p \equiv \text{T}$$

$$p \wedge \neg p \equiv \text{F}$$

$p$	$p \vee \neg p$	$p \wedge \neg p$
T	T	F
F	T	F

# Negation Intuition

$$p \vee \neg p \equiv \text{T}$$

$$p \wedge \neg p \equiv \text{F}$$

It is raining or it is not raining.

Always true

It is raining and it is not raining.

Always false

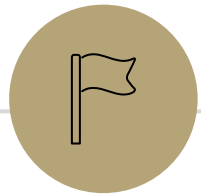
# Absorption

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

# Absorption

$p$	$q$	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	F	T	F
F	F	F	F	F	F



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## Logical Equivalence Examples

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Ex 1: Prove  $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv p \rightarrow q$

$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv ((p \vee \neg p) \wedge q) \vee (\neg p \wedge \neg q)$	Distributivity
$\equiv (T \wedge q) \vee (\neg p \wedge \neg q)$	Negation
$\equiv q \vee (\neg p \wedge \neg q)$	Identity
$\equiv (q \vee \neg p) \wedge (q \vee \neg q)$	Distributivity
$\equiv (q \vee \neg p) \wedge T$	Negation
$\equiv q \vee \neg p$	Identity
$\equiv \neg p \vee q$	Commutativity
$\equiv p \rightarrow q$	L.O.I

# Caveat 1: Associativity & Commutativity

Show that  $(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$ , following rules exactly.

$(p \vee q) \vee (r \vee s) \equiv p \vee (q \vee (r \vee s))$	Associativity
$\equiv p \vee (q \vee (s \vee r))$	Commutativity
$\equiv p \vee ((q \vee s) \vee r)$	Associativity
$\equiv ((q \vee s) \vee r) \vee p$	Commutativity
$\equiv (r \vee (q \vee s)) \vee p$	Commutativity
$\equiv r \vee (q \vee s) \vee p$	Order of Operations

# Caveat 1: Associativity & Commutativity

Show that  $(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$ .

We will allow abbreviated associativity & commutativity steps.

$$(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$$

Associativity & Commutativity

# Caveat 1: Associativity & Commutativity

Show that  $\neg p \vee p \equiv \text{T}$ .

Showing all steps:

$$\begin{aligned}\neg p \vee p &\equiv p \vee \neg p && \text{Commutativity} \\ &\equiv \text{T} && \text{Negation}\end{aligned}$$

What we allow:

$$\neg p \vee p \equiv \text{T} \quad \text{Negation}$$

## Caveat 2: Applying a rule twice

Expand  $(p \rightarrow q) \vee (q \rightarrow r)$  using the Law of Implication.

Showing all steps:

$$\begin{aligned}(p \rightarrow q) \vee (q \rightarrow r) &\equiv (\neg p \vee q) \vee (q \rightarrow r) && \text{Law of Implication} \\ &\equiv (\neg p \vee q) \vee (\neg q \vee r) && \text{Law of Implication}\end{aligned}$$

What we allow:

$$(p \rightarrow q) \vee (q \rightarrow r) \equiv (\neg p \vee q) \vee (\neg q \vee r) \quad \text{Law of Implication (x2)}$$

## Caveat 3: Applying rules to any proposition

We can apply equivalence rules to **any** proposition.

- $(p \vee q \wedge r) \vee (p \vee q \wedge r) \equiv p \vee q \wedge r$       Idempotency
- $\neg\neg(r \rightarrow \neg q) \equiv r \rightarrow \neg q$       Double Negation
- $\neg((p \vee q) \wedge s) \equiv \neg(p \vee q) \vee \neg s$       DeMorgan's Law

# Tautology, Contradiction, & Contingency

Definition:

A proposition is a **tautology** if it is always true.

E.g.  $p \vee \neg p$

A proposition is a **contradiction** if it is always false.

E.g.  $p \oplus p$

A proposition is a **contingency** if it can be either true or false.

E.g.  $p \vee q$

## Ex 2: Show $(p \wedge q) \rightarrow (q \vee p)$ is a tautology

$$(p \wedge q) \rightarrow (q \vee p) \equiv \neg(p \wedge q) \vee (q \vee p)$$

Law of Implication

$$\equiv (\neg p \vee \neg q) \vee (q \vee p)$$

DeMorgan's Law

$$\equiv (q \vee \neg q) \vee (p \vee \neg p)$$

Associativity & Commutativity

$$\equiv T \vee T$$

Negation (x2)

$$\equiv T$$

Domination (or Idempotency)

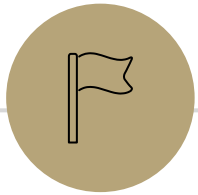
Ex 3: Simplify  $\neg p \rightarrow ((s \wedge p) \vee (\neg s \wedge p))$

Poll Everywhere  
**pollev.com/anjalia**

Not counted for points!

### Ex 3: Simplify $\neg p \rightarrow ((s \wedge p) \vee (\neg s \wedge p))$

$\neg p \rightarrow ((s \wedge p) \vee (\neg s \wedge p)) \equiv \neg p \rightarrow ((s \vee \neg s) \wedge p)$	Distributivity
$\equiv \neg p \rightarrow (T \wedge p)$	Negation
$\equiv \neg p \rightarrow p$	Identity
$\equiv \neg \neg p \vee p$	Law of Implication
$\equiv p \vee p$	Double Negation
$\equiv p$	Idempotency



# Normal Forms

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# Normal Forms

Given any truth table, can we create a propositional logic expression that generates that truth table?

$p$	$q$	$F(p, q)$
T	T	T
T	F	F
F	T	T
F	F	F

# Disjunctive Normal Form (DNF)

ORs of ANDs

1. Read the true rows of the truth table
2. AND together all settings in a true row
3. OR together the true rows

$p$	$q$	$F(p, q)$
T	T	T
T	F	F
F	T	T
F	F	F

$$F(p, q) \equiv (p \wedge q) \vee (\neg p \wedge q)$$

# Conjunctive Normal Form (CNF)

ANDs of ORs

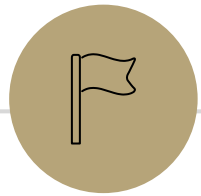
1. Read the false rows of the truth table
2. OR together **the negation** of all settings in a false row
3. AND together the false rows

$p$	$q$	$F(p, q)$
T	T	T
T	F	<b>F</b>
F	T	T
F	F	<b>F</b>

$$F(p, q) \equiv (\neg p \vee q) \wedge (p \vee q)$$

# Normal Forms

- Don't simplify CNF / DNF further
- These are standard forms – everyone's CNF / DNF formulas will be the same (up to commutativity)



# Boolean Algebra & Circuits

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# Notation

- Logic is fundamental
  - Computer scientists use it in programs
  - Mathematicians use it in proofs
  - Engineers use it in hardware
  - Philosophers use it in arguments
- Consequently, everyone has their own notation

# Boolean Algebra

Another notation for logic. Preferred by some because it's compact.

Term	Propositional Logic	Boolean Algebra
or	$\vee$	$+$
and	$\wedge$	$\cdot$
not	$\neg$	$'$
True	<b>T</b>	<b>1</b>
False	<b>F</b>	<b>0</b>

$(p \wedge q \wedge r) \vee s \vee \neg t$  in Boolean Algebra is  $pqr + s + t'$ .

# Digital Circuits

## Computing with Logic

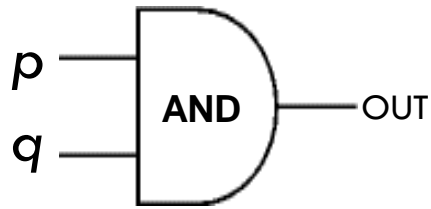
- T corresponds to 1, or high voltage
- F corresponds to 0, or low voltage

## Gates

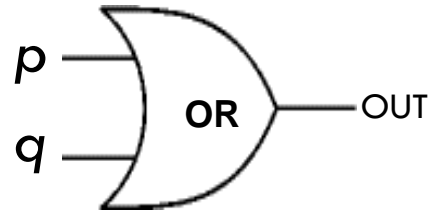
- Gates take inputs and produce outputs

# Digital Circuits – Gates

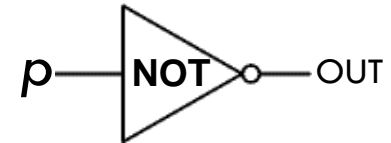
AND Gate



OR Gate



NOT Gate



# Digital Circuits – Example

Write  $\neg p \wedge (\neg q \wedge (r \vee s))$  as a circuit.

