Equivalences, Normal Forms

CSE 311: Foundations of Computing I
Lecture 3

Announcements

- HW1
  - Due Wednesday, 11:59 pm
  - Submit on Gradescope
- Office Hours begin today

Review

Propositions

Propositions

- Propositions are T/F-valued variables
- Combine using logical connectives ($\neg, \vee, \wedge, \rightarrow$,etc.)
- Can be described by a truth table

Applications

- Understanding complex English sentences
- Simplifying Boolean logic in code

Logical Equivalence

$A \equiv B$ is an assertion that two propositions A, B always have the same truth values.

There are two generic methods to proving equivalence:

1. Make a truth table for both propositions
2. Use a chain of logical equivalences

Recall: Equivalence Laws

Double Negation

$\neg\neg p \equiv p$

DeMorgan's Laws

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Law of Implication

$p \to q \equiv \neg p \vee q$

Contrapositive

$p \to q \equiv \neg q \to \neg p$

Commutativity

$p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

Associativity

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Exercise

Prove that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ using the logical equivalences rules we've discussed so far.

Do not use contrapositive in the proof.

$\neg q \rightarrow \neg p \equiv \neg\neg q \vee \neg p$	Law of Implication
$\equiv q \vee \neg p$	Double Negation
$\equiv \neg p \vee q$	Commutativity
$\equiv p \rightarrow q$	Law of Implication

Logical Equivalences Cont.

Distributivity

$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$

Distributivity: Intuition

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

You go to class, and you read the notes or the textbook.

You go to class and read the notes, or you go to class and you read the textbook.

Identity

$$p \wedge \mathrm{T} \equiv p$$

$$p \vee \mathrm{F} \equiv p$$

p	$p \wedge \mathrm{T}$	$p \vee \mathrm{F}$
T	T	T
F	F	F

Domination

$$p \vee \mathrm{T} \equiv \mathrm{T}$$

$$p \wedge \mathrm{F} \equiv \mathrm{F}$$

p	$p \vee \mathrm{T}$	$p \wedge \mathrm{F}$
T	T	F
F	T	F

Idempotency

$$p \vee p \equiv p$$

$$p \wedge p \equiv p$$

p	$p \vee p$	$p \wedge p$
T	T	T
F	F	F

Negation

$$p \lor \neg p \equiv \mathrm{T}$$

$$p \land \neg p \equiv \mathrm{F}$$

p	$p \lor \neg p$	$p \land \neg p$
T	T	F
F	T	F

Negation Intuition

$$p \vee \neg p \equiv \mathrm{T}$$

$$p \wedge \neg p \equiv \mathrm{F}$$

It is raining or it is not raining. Always true

It is raining and it is not raining. Always false

Absorption

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

Absorption

Exercise: Build the truth tables to confirm.

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T	T				
T	F				
F	T				
F	F				

Logical Equivalence Examples

Ex 1: Prove $(p \land q) \lor (\neg p \land q) \lor (\neg p \land \neg q) \equiv p \to q$

Caveat 1: Associativity & Commutativity

Show that $(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$, following rules exactly.

Caveat 1: Associativity & Commutativity

Show that $(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$.

We will allow abbreviated associativity & commutativity steps.

$(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$ Associativity & Commutativity

Caveat 1: Associativity & Commutativity

Show that $\neg p \lor p \equiv \mathrm{T}$.

Showing all steps:

What we allow:

Caveat 2: Applying a rule twice

Expand $(p \rightarrow q) \vee (q \rightarrow r)$ using the Law of Implication.

Showing all steps:

What we allow:

Caveat 3: Applying rules to any proposition

We can apply equivalence rules to **any** proposition.

- $(p \vee q \wedge r) \vee (p \vee q \wedge r) \equiv$ ______________ ___________________
- $\neg\neg(r \rightarrow \neg q) \equiv$ ___________________________ ___________________
- $\neg\big((p \vee q) \wedge s\big) \equiv$ _______________________ ___________________

Tautology, Contradiction, & Contingency

Definition:

A proposition is a tautology if ________________________________.

E.g. _______________

A proposition is a contradiction if ______________________________.

E.g. _______________

A proposition is a contingency if ______________________________.

E.g. _______________

Ex 2: Show $(p \land q) \to (q \lor p)$ is a tautology

Ex 3: Simplify $\neg p \rightarrow \big((s \wedge p) \vee (\neg s \wedge p)\big)$

Poll Everywhere

pollev.com/anjalia

Not counted for points!

Ex 3: Simplify $\neg p \to \big((s \wedge p) \vee (\neg s \wedge p)\big)$

$\neg p \to \big((s \wedge p) \vee (\neg s \wedge p)\big)$	$\equiv \neg p \to \big((s \vee \neg s) \wedge p\big)$	Distributivity
	$\equiv \neg p \to (\mathrm{T} \wedge p)$	Negation
	$\equiv \neg p \to p$	Identity
	$\equiv \neg\neg p \vee p$	Law of Implication
	$\equiv p \vee p$	Double Negation
	$\equiv p$	Idempotency

Normal Forms

Normal Forms

Given any truth table, can we create a propositional logic expression that generates that truth table?

p	q	$F(p,q)$
T	T	T
T	F	F
F	T	T
F	F	F

Disjunctive Normal Form (DNF)

ORs of ANDs

1. Read the true rows of the truth table
2. AND together all settings in a true row
3. OR together the true rows

p	q	$F(p,q)$
T	T	T
T	F	F
F	T	T
F	F	F

Conjunctive Normal Form (CNF)

ANDs of ORs

1. Read the false rows of the truth table
2. OR together **the negation** of all settings in a false row
3. AND together the false rows

p	q	$F(p,q)$
T	T	T
T	F	F
F	T	T
F	F	F

Normal Forms

- Don’t simplify CNF / DNF further
- These are standard forms – everyone's CNF / DNF formulas will be the same (up to commutativity)

Boolean Algebra & Circuits

Notation

- Logic is fundamental
  - Computer scientists use it in programs
  - Mathematicians use it in proofs
  - Engineers use it in hardware
  - Philosophers use it in arguments
- Consequently, everyone has their own notation

Boolean Algebra

Another notation for logic. Preferred by some because it's compact.

Term	Propositional Logic	Boolean Algebra
or	$\vee$	$+$
and	$\wedge$	$\cdot$
not	$\neg$	$'$
True	$\mathbf{T}$	1
False	$\mathbf{F}$	0

$(p \wedge q \wedge r) \vee s \vee \neg t$ in Boolean Algebra is ______________________.

Digital Circuits

Computing with Logic

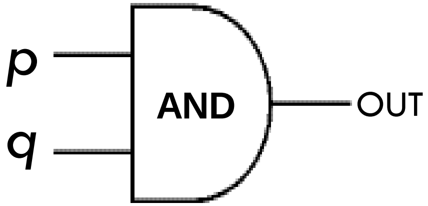
- $\mathbf{T}$ corresponds to $\mathbf{1}$, or high voltage
- $\mathbf{F}$ corresponds to $\mathbf{0}$, or low voltage

Gates

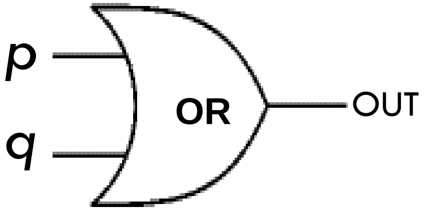
- Gates take inputs and produce outputs

Digital Circuits – Gates

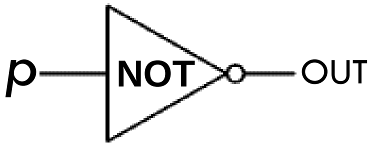
AND Gate



OR Gate



NOT Gate



Digital Circuits – Example

Write $\neg p \land (\neg q \land (r \lor s))$ as a circuit.