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Equivalences, Normal Forms

CSE 311: Foundations of Computing I
Lecture 3

Announcements

- HW1
  - Due Wednesday, 11:59 pm
  - Submit on Gradescope
- Office Hours begin today

Review

Propositions

Propositions

- Propositions are T/F-valued variables
- Combine using logical connectives ($\neg, \vee, \wedge, \rightarrow$, etc.)
- Can be described by a truth table

Applications

- Understanding complex English sentences
- Simplifying Boolean logic in code

Logical Equivalence

$A \equiv B$ is an assertion that two propositions A, B always have the same truth values.

There are two generic methods to proving equivalence:

1. Make a truth table for both propositions
2. Use a chain of logical equivalences

$$
\begin{aligned}
A &\equiv \underline{\qquad\qquad} \quad \langle \quad \rangle \\
&\equiv \underline{\qquad\qquad} \quad \langle \quad \rangle \\
&\equiv \underline{\qquad\qquad} \quad \langle \quad \rangle \\
&\equiv B
\end{aligned}
$$

A	B
–	–
–	–
–	–

n

2^n

Recall: Equivalence Laws

Double Negation

$\neg\neg p \equiv p$

DeMorgan's Laws

1 $\neg(p \vee q) \equiv \neg p \wedge \neg q$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Law of Implication

$p \rightarrow q \equiv \neg p \vee q$

Contrapositive

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Commutativity

$p \vee q \equiv q \vee p$ $\quad p + q = q + p$

$p \wedge q \equiv q \wedge p$ $\quad p \cdot q = q \cdot p$

Associativity

$(p + q) + r$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $\quad = p + (q + r)$

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Exercise

Prove that $p \to q \equiv \neg q \to \neg p$ using the logical equivalences rules we've discussed so far.

Do not use contrapositive in the proof.

$$
\begin{aligned}
\neg q \to \neg p &\equiv \neg\neg q \vee \neg p && \text{Law of Implication} \\
&\equiv q \vee \neg p && \text{Double Negation} \\
&\equiv \neg p \vee q && \text{Commutativity} \\
&\equiv p \to q && \text{Law of Implication}
\end{aligned}
$$

Logical Equivalences Cont.

Distributivity

$p \cdot (q + r) = p \cdot q + p \cdot r$

$$p \wedge (q \vee r) \equiv \underline{(p \wedge q) \vee (p \wedge r)}$$

$$p \vee (q \wedge r) \equiv \underline{(p \vee q)} \wedge \underline{(p \vee r)}$$

Distributivity: Intuition

p: you go to class

q: you read the notes

r: you read the textbook

1 $$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

You go to class, and you read the notes or the textbook. $p \land (q \lor r)$

You go to class and read the notes, or you go to class and you read the textbook. $(p \land q) \lor (p \land r)$

Identity

flag

$$p \wedge \mathrm{T} \equiv p$$

$$p \vee \mathrm{F} \equiv p$$

if(flag && true)

if(flag)

if(flag || false)

p	$p \wedge \mathrm{T}$	$p \vee \mathrm{F}$
T	T	T
F	F	F

Domination

~~if(flag && false){~~
~~<code>~~
~~}~~

$$p \vee \underline{\mathrm{T}} \equiv \mathrm{T}$$

$$p \wedge \underline{\mathrm{F}} \equiv \mathrm{F}$$

if(flag || true){
<code>
}

<code>

p	$p \vee \mathrm{T}$	$p \wedge \mathrm{F}$
T	T	F
F	T	F

Idempotency

$$p \vee p \equiv p$$

$$p \wedge p \equiv p$$

p	$p \vee p$	$p \wedge p$
T	T	T
F	F	F

Negation

$$p \vee \neg p \equiv \mathrm{T}$$

$$p \wedge \neg p \equiv \mathrm{F}$$

p	$p \vee \neg p$	$p \wedge \neg p$
T	T	F
F	T	F

Negation Intuition

$$p \vee \neg p \equiv \mathrm{T}$$

$$p \wedge \neg p \equiv \mathrm{F}$$

It is raining or it is not raining. Always true

It is raining and it is not raining. Always false

Absorption

$$p \lor (p \land q) \equiv p$$

$$p \land (p \lor q) \equiv p$$

① cases:

case where p is T. Then $p \lor (p \land q)$

becomes $T \lor (T \land q)$, which is T.

case where p is F. Then $p \lor (p \land q)$

becomes $F \lor (F \land q)$, which is F.

Absorption

Exercise: Build the truth tables to confirm.

p	q	$p \land q$	$p \lor (p \land q)$	$p \lor q$	$p \land (p \lor q)$
T	T				
T	F				
F	T				
F	F				

Logical Equivalence Examples

Ex 1: Prove $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv p \rightarrow q$

$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv ((p \vee \neg p) \wedge q) \vee (\neg p \wedge \neg q)$ Dist.

$\equiv (T \wedge q) \vee (\neg p \wedge \neg q)$ Negation

$\equiv q \vee (\neg p \wedge \neg q)$ identity

$\equiv (q \vee \neg p) \wedge (q \vee \neg q)$ Dist.

$\equiv (q \vee \neg p) \wedge T$ Negation

$\equiv q \vee \neg p$ identity

$\equiv \neg p \vee q$ Commutativity

$\equiv p \rightarrow q$ Law of Implication

Tips:

* Group propositions w/ themselves (or their negations)
* Simplify all T/F
* Work forwards & backwards

Caveat 1: Associativity & Commutativity

Show that $\underset{\sim}{(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s)} \vee p$, following rules exactly.

$(p \vee q) \vee (r \vee s) \equiv p \vee (q \vee (\underline{r \vee s}))$ Assoc.

$\equiv p \vee (q \vee (s \vee r))$ Commut.

$\equiv p \vee ((q \vee s) \vee r)$ Assoc.

$\equiv ((q \vee s) \vee r) \vee p$ Commut.

$\equiv (r \vee (q \vee s)) \vee p$ Commut.

$\equiv r \vee (q \vee s) \vee p$ order of op.

Caveat 1: Associativity & Commutativity

Show that $(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$.

We will allow abbreviated associativity & commutativity steps.

$(p \vee q) \vee (r \vee s) \equiv r \vee (q \vee s) \vee p$ Associativity & Commutativity

Caveat 1: Associativity & Commutativity

Show that $\underline{\neg p \vee p \equiv} \mathrm{T}$.

Showing all steps:

$\neg p \vee p \equiv p \vee \neg p$ Commut.

$\equiv T$ Negation

What we allow:

$\neg p \vee p \equiv T$ Negation

Caveat 2: Applying a rule twice

Expand $(p \to q) \vee (q \to r)$ using the Law of Implication.

Showing all steps:

$$(p \to q) \vee (q \to r) \equiv (\neg p \vee q) \vee (q \to r) \quad \text{L.O.I}$$

$$\equiv (\neg p \vee q) \vee (\neg q \vee r) \quad \text{L.O.I.}$$

What we allow:

$$(p \to q) \vee (q \to r) \equiv (\neg p \vee q) \vee (\neg q \vee r)$$

<u>L.O.I (x2)</u>

Caveat 3: Applying rules to any proposition

We can apply equivalence rules to **any** proposition.

- $(p \vee q \wedge r) \vee (p \vee q \wedge r) \equiv$ $p \vee q \wedge r$ — idempotency
- $\neg\neg(r \rightarrow \neg q) \equiv$ $r \rightarrow \neg q$ — Double Negation
- $\neg\big((p \vee q) \wedge s\big) \equiv$ $\neg(p \vee q) \vee \neg s$ — De Morgan's Law

$\equiv (\neg p \wedge \neg q) \vee \neg s$

Tautology, Contradiction, & Contingency

Definition:

A proposition is a **tautology** if it is always true.

E.g. $p \vee \neg p$

A proposition is a **contradiction** if it is always false.

E.g. $p \wedge \neg p$, $p \oplus p$

A proposition is a **contingency** if it can be either T or F.

E.g. $p \vee q$

Ex 2: Show $(p \wedge q) \to (q \vee p)$ is a tautology

$$(p \wedge q) \to (q \vee p) \equiv \neg(p \wedge q) \vee (q \vee p) \quad \text{L.O.I.}$$

$$\equiv (\neg p \vee \neg q) \vee (q \vee p) \quad \text{DeMorgan's}$$

$$\equiv (p \vee \neg p) \vee (q \vee \neg q) \quad \text{Commute \& Assoc.}$$

$$\equiv T \vee T \quad \text{Negation}$$

$$\equiv T \quad \text{Domination (idempotency)}$$

* Use L.O.I. to get rid of $\to$

* Group like vars together

Ex 3: Simplify $\neg p \rightarrow ((s \wedge p) \vee (\neg s \wedge p))$

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pollev.com/anjalia

Not counted for points!

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Ex 3: Simplify $\neg p \rightarrow \big((s \wedge p) \vee (\neg s \wedge p)\big)$

$$\begin{aligned}
\neg p \rightarrow \big((s \wedge p) \vee (\neg s \wedge p)\big) &\equiv \neg p \rightarrow \big((s \vee \neg s) \wedge p\big) && \text{Distributivity} \\
&\equiv \neg p \rightarrow (\mathrm{T} \wedge p) && \text{Negation} \\
&\equiv \neg p \rightarrow p && \text{Identity} \\
&\equiv \neg\neg p \vee p && \text{Law of Implication} \\
&\equiv p \vee p && \text{Double Negation} \\
&\equiv p && \text{Idempotency}
\end{aligned}$$

Normal Forms

Normal Forms

Given any truth table, can we create a propositional logic expression that generates that truth table?

$\boxed{q}$

p	q	$F(p,q)$
T	T	T
T	F	F
F	T	T
F	F	F

$\boxed{q \vee (p \wedge q)}$

$(\neg p \vee q) \wedge (p \vee q)$

Disjunctive Normal Form (DNF)

ORs of ANDs

1. Read the true rows of the truth table
2. AND together all settings in a true row
3. OR together the true rows

p	q	$F(p, q)$
T	T	T
T	F	F
F	T	T
F	F	F

$\leftarrow p \wedge q$

$\leftarrow \neg p \wedge q$

$\boxed{(p \wedge q) \vee (\neg p \wedge q)}$

$\equiv q$

Conjunctive Normal Form (CNF)

ANDs of ORs

1. Read the false rows of the truth table
2. OR together **the negation** of all settings in a false row
3. AND together the false rows

$$\neg((p \wedge \neg q) \vee (\neg p \wedge \neg q))$$
$$\equiv \neg(p \wedge \neg q) \wedge \neg(\neg p \wedge \neg q)$$
$$\equiv (\neg p \vee q) \wedge (p \vee q)$$

p	q	$F(p,q)$	
T	T	T	F
T	F	F	T
F	T	T	F
F	F	F	T

$$* \boxed{(\neg p \vee q) \wedge (p \vee q)}$$

Normal Forms

- Don’t simplify CNF / DNF further
- These are standard forms – everyone's CNF / DNF formulas will be the same (up to commutativity)

Boolean Algebra & Circuits

Notation

- Logic is fundamental
  - Computer scientists use it in programs
  - Mathematicians use it in proofs
  - Engineers use it in hardware
  - Philosophers use it in arguments
- Consequently, everyone has their own notation

Boolean Algebra

Another notation for logic. Preferred by some because it's compact.

Term	Propositional Logic	Boolean Algebra
or	$\vee$	$+$
and	$\wedge$	$\cdot$
not	$\neg$	$'$
True	T	1
False	F	0

$\neg p \quad p'$

$(p \wedge q \wedge r) \vee s \vee \neg t$ in Boolean Algebra is $p \cdot q \cdot r + s + t'$.

$pqr + s + t'$

Digital Circuits

Computing with Logic

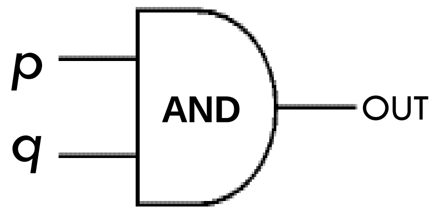
- $\mathbf{T}$ corresponds to $\mathbf{1}$, or high voltage
- $\mathbf{F}$ corresponds to $\mathbf{0}$, or low voltage

Gates

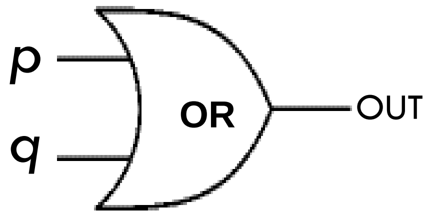
- Gates take inputs and produce outputs

Digital Circuits – Gates

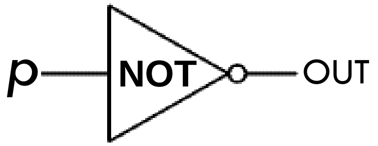
AND Gate



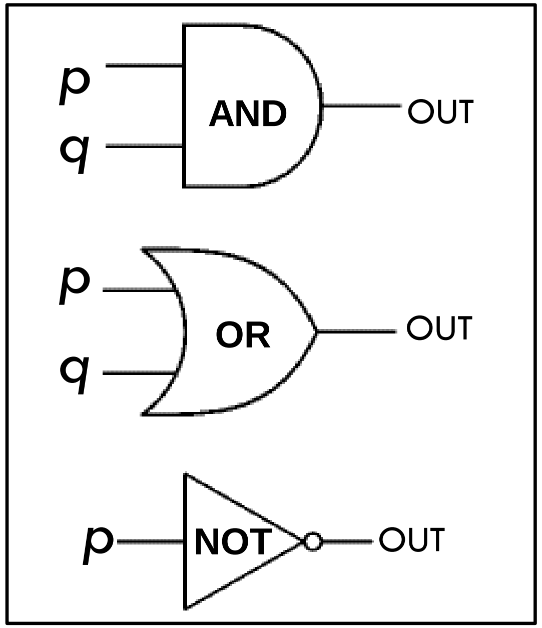
OR Gate



NOT Gate



Digital Circuits – Example



Write $\neg p \wedge \left(\neg q \wedge (r \vee s)\right)$ as a circuit.