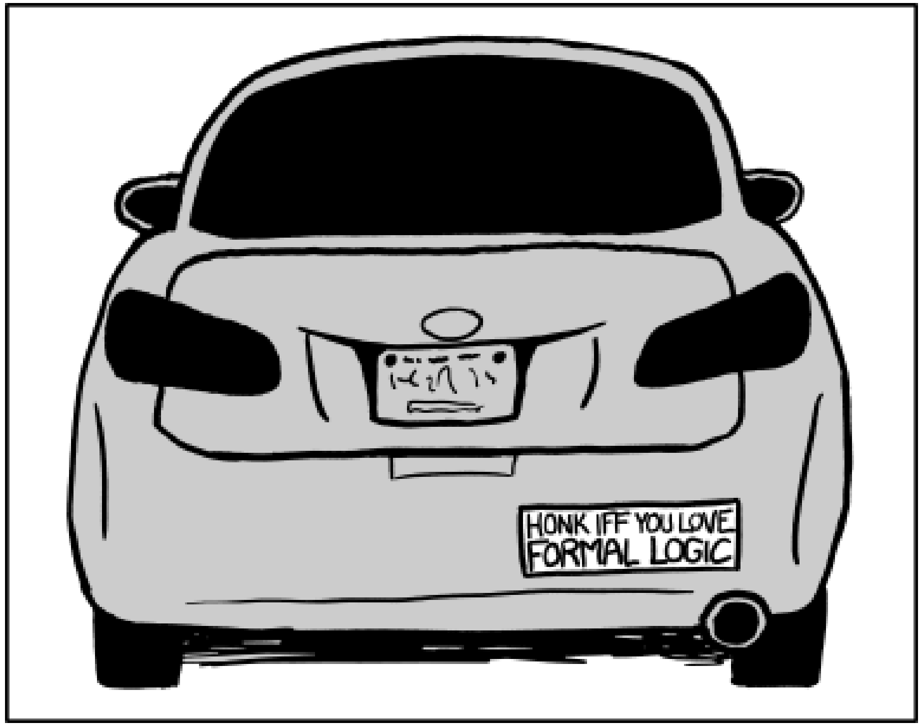

Propositional Logic, Equivalences

CSE 311: Foundations of Computing I
Lecture 2

Announcements

- HW1 posted on the course website under assignments
  - Due Wednesday, 11:59 pm
  - Late Policy applies, late due Friday, 11:59 pm
  - Submit on Gradescope
- OH begin on Monday

Review

Recall: Atomic Propositions

- **Atomic Propositions** are true or false statements that cannot be broken down any further
- Propositional variables: p, q, r, s ...

Recall: Logical Connectives

Name	Logical Symbol
Not	$\neg p$
And	$p \wedge q$
Or	$p \vee q$
XOR	$p \oplus q$
Implication	$p \rightarrow q$
Biconditional	$p \leftrightarrow q$

Recall: Truth Tables

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Recall: Implication

"If it's raining, then I have my umbrella"

p: It is raining q: I have my umbrella

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalently:

- Whenever it is raining, I have my umbrella.
- It is raining only if I have my umbrella.
- For it to be raining, it is necessary that I have my umbrella.

Translating Propositions Cont.

Compound Proposition Example

Unless I go to a café or to campus, I do not drink coffee, but also I don't go to cafés.

What does this mean? Find the atomic propositions and translate to logic.

p: I go to a café

q: I go to campus

r: I drink coffee

$$(\neg(p \vee q) \rightarrow \neg r) \wedge (\neg p)$$

Compound Proposition Example

Unless I go to a café or to campus, I do not drink coffee, but also I don’t go to cafés.

p: I go to a café

q: I go to campus

r: I drink coffee

$$(\neg(p \vee q) \rightarrow \neg r) \wedge (\neg p)$$

When is this true? When is this false? Let’s construct a truth table.

Compound Proposition Example

p	q	r	$p \lor q$	$\neg(p \lor q)$	$\neg r$	$\neg(p \lor q) \rightarrow \neg r$	$\neg p$	$(\neg(p \lor q) \rightarrow \neg r) \land (\neg p)$
T	T	T	T	F	F	T	F	F
T	T	F	T	F	T	T	F	F
T	F	T	T	F	F	T	F	F
T	F	F	T	F	T	T	F	F
F	T	T	T	F	F	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	F	T	F	F	T	F
F	F	F	F	T	T	T	T	T

Recall: Logical Connectives

Name	Logical Symbol
Not	$\neg p$
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Implication	$p \rightarrow q$
Biconditional	$p \leftrightarrow q$

Biconditional ($\leftrightarrow$)

$p \leftrightarrow q$ is true when p, q have the same truth values, and false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional ($\leftrightarrow$)

Translations of $p \leftrightarrow q$ include:

- p if and only if q
- p iff q
- p is equivalent to q
- p implies q, and q implies p
- If p then q, and if q then p

Biconditional ($\leftrightarrow$)

Why is this a proposition? Can be true or false.

p: I am a legal adult

q: I am 18 years or older

r: I am 25 years or older

$p \leftrightarrow q$ is true

$p \leftrightarrow r$ is false

Biconditional ($\leftrightarrow$)

$p \leftrightarrow q$ has the same truth table as $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Order of Operations

( )	Parentheses
$\neg$	Negation
$\wedge$	And
$\vee$ $\oplus$	Or, XOR
$\rightarrow$	Implication
$\leftrightarrow$	Biconditional

Within the same order, evaluate from left to right.

$p \vee \neg q \rightarrow r$ is the same as $\big(p \vee (\neg q)\big) \rightarrow r$.

Logical Equivalence

Logical Equivalence

Definition:

Two propositions are logically equivalent if they have identical truth values.

The notation for A and B being logically equivalent is $\mathrm{A} \equiv \mathrm{B}$.

Examples:

$p \vee q \equiv q \vee p$

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$=$ vs. $\equiv$ vs. $\leftrightarrow$

$\mathrm{A} = \mathrm{B}$ means A, B are the exact same string.

E.g. $p \vee q = p \vee q$, but $p \vee q \neq q \vee p$. We hardly use $=$ with propositions!

$\mathrm{A} \equiv \mathrm{B}$ is an assertion that A, B always have the same truth value.

E.g. $p \vee q \equiv q \vee p$.

$\mathrm{A} \leftrightarrow \mathrm{B}$ is a proposition that might be true or false.

E.g. $p \vee q \leftrightarrow p \wedge q$ is a false proposition.

$A \equiv B$ has the same meaning as $(A \leftrightarrow B) \equiv T$.

Proving Logical Equivalence

Motivation

Given two propositions, we would like to know if they are equivalent.

E.g. one developer wrote $\textbf{if}\left((p \;\&\&\; p) \;||\; p\right) \{\ldots\}$.

Another developer wrote $\textbf{if}\,(p)\;\{\ldots\}$.

You want to confirm if those are the same.

Given a complicated proposition, we would like to find a simpler proposition that it's equivalent to.

E.g. you see $\textbf{if}\left((p \;\&\&\; p) \;||\; p\right)\{\ldots\}$ in code.

You simplify the code to $\textbf{if}\,(p)\;\{\ldots\}$.

Strategy 1: Truth Tables

Make a truth table for the two propositions and check if they are the same.

p vs. $(p \wedge p) \vee p$

p	$p \wedge p$	$(p \wedge p) \vee p$
T	T	T
F	F	F

Strategy 1: Truth Tables

Truth tables **do** let us check if two propositions are equivalent.

Truth tables **don't** give us a good way to start from a complicated proposition, and simplify it.

Strategy 2: Manipulating Expressions

Instead, we are going to learn **logical equivalence rules** to help us simplify expressions.

Similar to algebra, where we can apply rules to transform expression:

$(x + 2)(x + 3) = x^2 + 2x + 3x + 6$ Distributivity

$= x^2 + 5x + 6$ Adding like terms

For each rule, we will **understand why it's true**, and **practice** using it.

Logical Equivalence Rules

Double Negation

$$\neg\neg p \equiv p$$

I am not not loving propositional logic.

I am loving propositional logic.

De Morgan's Laws: Intuition

Consider the following sentences:

- I don't like apples or mangoes. $\neg(p \vee q)$
- I don't like apples, and I don't like mangoes. $\neg p \wedge \neg q$

Are they logically equivalent? Intuitively, yes

De Morgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

De Morgan’s Laws

Example: $\neg(p \lor q) \equiv \neg p \land \neg q$

p	q	$p \lor q$	$\neg(p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

De Morgan’s Laws

```
if (!(front != null && value > front.data)) {…}

if (front == null || value <= front.data) {…}
```

Law of Implication

Implications are unusual. Can we write them using ANDs ORs & NOTs?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Law of Implication

$$p \to q \equiv \neg p \vee q$$

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Law of Implication: Intuition

$$p \to q \equiv \neg p \vee q$$

If it is raining, then I have my umbrella.

It is not raining, or I have my umbrella.

Converse & Contrapositive

Implication: $p \to q$ | If it's raining, I have my umbrella.

Converse: $q \to p$ | If I have my umbrella, it's raining.

Contrapositive: $\neg q \to \neg p$ | If I don't have my umbrella, it's not raining.

Converse & Contrapositive

Implication: $p \to q$ Converse: $q \to p$ Contrapositive: $\neg q \to \neg p$

How do these relate?

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	T	T	F	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Contrapositive: Intuition

$$p \to q \equiv \neg q \to \neg p$$

If an animal is a cat, then it is a mammal.

If an animal is not a mammal, then it's not a cat.

Commutativity

$$p \lor q \equiv q \lor p$$

$$p \land q \equiv q \land p$$

It is raining or it is June.

It is June or it is raining.

Associativity

$$(p \lor q) \lor r \equiv p \lor (q \lor r)$$

$$p \land (q \land r) \equiv p \land (q \land r)$$

They perform at 3:00 and 5:00, and also 8:00.

They perform at 3:00, and also 5:00 and 8:00.

WARNING

Only apply associativity when all connectives are AND, or all connectives are OR

Exercise

Prove that $p \to q \equiv \neg q \to \neg p$ using the logical equivalences rules we've discussed so far.

Do not use contrapositive in the proof.

$\neg q \to \neg p \equiv \neg\neg q \vee \neg p$	Law of Implication
$\equiv q \vee \neg p$	Double Negation
$\equiv \neg p \vee q$	Commutativity
$\equiv p \to q$	Law of Implication

Distributivity

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Distributivity: Intuition

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

You go to class, and you read the notes or the textbook.

You go to class and read the notes, or you go to class and you read the textbook.

Identity

$$p \wedge \mathrm{T} \equiv p$$

$$p \vee \mathrm{F} \equiv p$$

p	$p \wedge \mathrm{T}$	$p \vee \mathrm{F}$
T	T	T
F	F	F

Domination

$$p \lor \mathrm{T} \equiv \mathrm{T}$$

$$p \land \mathrm{F} \equiv \mathrm{F}$$

p	$p \lor \mathrm{T}$	$p \land \mathrm{F}$
T	T	F
F	T	F

Idempotency

$$p \vee p \equiv p$$

$$p \wedge p \equiv p$$

p	$p \vee p$	$p \wedge p$
T	T	T
F	F	F

Negation

$$p \lor \neg p \equiv \mathrm{T}$$

$$p \land \neg p \equiv \mathrm{F}$$

p	$p \lor \neg p$	$p \land \neg p$
T	T	F
F	T	F

Negation Intuition

$$p \lor \neg p \equiv \mathrm{T}$$

$$p \land \neg p \equiv \mathrm{F}$$

It is raining or it is not raining.	Always true
It is raining and it is not raining.	Always false

Absorption

$$p \lor (p \land q) \equiv p$$

$$p \land (p \lor q) \equiv p$$

Exercise: Build the truth tables to confirm.

Absorption

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	F	T	F
F	F	F	F	F	F