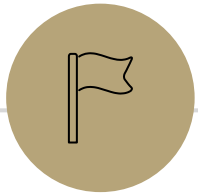


Propositional Logic, Equivalences

CSE 311: Foundations of
Computing I
Lecture 2

Announcements

- HW1 posted on the course website under assignments
 - Due Wednesday, 11:59 pm
 - Late Policy applies, late due Friday, 11:59 pm
 - Submit on Gradescope
- OH begin on Monday



Review

Recall: Atomic Propositions

- Atomic Propositions are true or false statements that cannot be broken down any further
- Propositional variables: $p, q, r, s \dots$

P, Q, R, S, \dots

T - true

F - false

Recall: Logical Connectives

Name

Logical Symbol

Not

$\neg p$

And

$p \wedge q$

Or

$p \vee q$

XOR

$p \oplus q$

Implication

$p \rightarrow q$

Biconditional

$p \leftrightarrow q$

Recall: Truth Tables

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Recall: Implication

$p \rightarrow q$

✓ "If it's raining, then I have my umbrella"

p : It is raining q : I have my umbrella

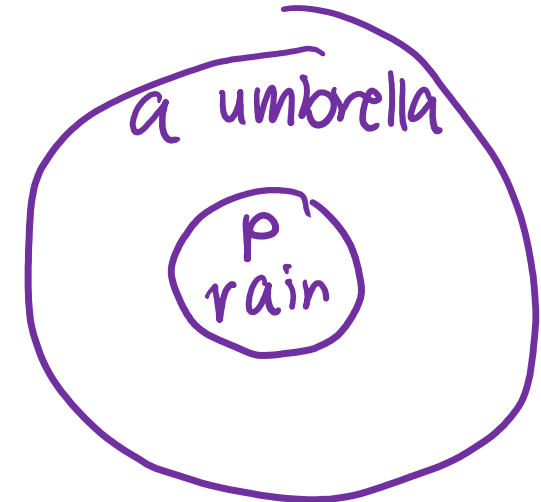
$p \rightarrow q$

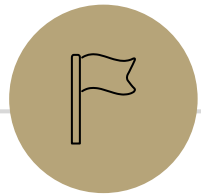
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

vacuous {

Equivalently:

- ✓ • Whenever it is raining, I have my umbrella.
- ✓ • It is raining only if I have my umbrella.
- ✓ • For it to be raining, it is necessary that I have my umbrella.





Translating Propositions Cont.

Compound Proposition Example

Unless I go to a café or to campus, I do not drink coffee, but also I don't go to cafés.

p : I go to a café f

q : I go to campus \top

r : I drink coffee \top

$$(\neg(p \vee q) \rightarrow \neg r) \wedge (\neg p)$$

When is this true? When is this false? Let's construct a truth table.

Compound Proposition Example

p	q	r	$p \vee q$	$\neg(p \vee q)$	$\neg r$	$\neg(p \vee q) \rightarrow \neg r$	$\neg p$	$(\neg(p \vee q) \rightarrow \neg r) \wedge (\neg p)$
T	T	T	T	F	F	T	F	F
T	T	F	T	F	T	T	F	F
T	F	T	T	F	F	T	F	F
T	F	F	T	F	T	T	F	F
F	T	T	T	F	F	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	F	T	F	F	T	F
F	F	F	F	T	T	T	T	T

Recall: Logical Connectives

Name

Logical Symbol

Not

$\neg p$

And

$p \wedge q$

Or

$p \vee q$

XOR

$p \oplus q$

Implication

$p \rightarrow q$

Biconditional

$p \leftrightarrow q$

Biconditional (\leftrightarrow)

$p \leftrightarrow q$ is true when p, q have the same truth values, and false otherwise.

"sameness"
"equivalence"

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional (\leftrightarrow)

Translations of $p \leftrightarrow q$ include:

- p if and only if q
- p iff q
- p is equivalent to q
- p implies q & q implies p
- if p then q & if q then p

$$\underline{(p \rightarrow q) \wedge (q \rightarrow p)}$$



Biconditional (\leftrightarrow)

Why is this a proposition? True/False value

p : I am a legal adult

q : I am 18 years or older

r : I am 25 years or older

$p \leftrightarrow q$ (T)

$p \leftrightarrow r$ (F)
T F

Biconditional (\leftrightarrow)

"p if and only if q"

$$\wedge \begin{matrix} q \rightarrow p \\ p \rightarrow q \end{matrix}$$

$p \leftrightarrow q$ has the same truth table as $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Order of Operations

$()$	Parentheses
\neg	Negation
\wedge	And
$\vee \oplus$	Or, XOR
\rightarrow	Implication
\leftrightarrow	Biconditional

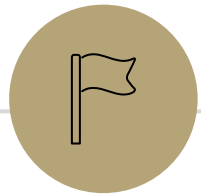
PEMDAS
↑ ↑ ∪ ∪
 $() a^x$

$p \vee q \wedge r$

$p \vee (q \wedge r)$

Within the same order, evaluate from left to right.

$p \vee \neg q \rightarrow r$ is the same as $(p \vee (\neg q)) \rightarrow r$.



Logical Equivalence

Logical Equivalence

Definition:

Two propositions are **logically equivalent** if they have identical truth tables.

The notation for A and B being logically equivalent is $A \equiv B$.

Examples:

$$\checkmark p \vee q \equiv q \vee p$$

$$\checkmark p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

= VS. \equiv VS. \leftrightarrow $A \equiv B$ means $(A \leftrightarrow B) \equiv T$

~~$A \neq B$ means A, B are the exact same string.~~

$$p \vee q = p \vee q \qquad p \vee q \neq q \vee p$$

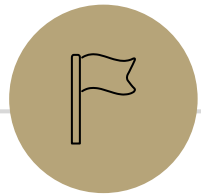
$A \equiv B$ is an assertion that A, B always have the same truth value.

$$p \vee q \equiv q \vee p \qquad p \vee q \neq p \wedge q$$

$A \leftrightarrow B$ is a proposition that might be true or false.

$$p \vee q \leftrightarrow q \vee p \quad (\text{T}) \qquad [p \vee q \leftrightarrow p \wedge q$$

F when p is T q is F



Proving Logical Equivalence



Motivation

Given two propositions, we would like to know if they are equivalent.

E.g. one developer wrote `if ((p && p) || p) { ... }`.

$$(p \wedge p) \vee p$$

Another developer wrote `if (p) { ... }`.

$$p$$

You want to confirm if those are the same.

Given a complicated proposition, we would like to find a simpler proposition that it's equivalent to.

E.g. you see `if ((p && p) || p) { ... }` in code.

You simplify the code to `if (p) { ... }`.

Strategy 1: Truth Tables

Make a truth table for the two propositions and check if they are the same.

p vs. $(p \wedge p) \vee p$

p	$p \wedge p$	$(p \wedge p) \vee p$
T	T	T
F	F	F

Strategy 1: Truth Tables

- ✓ Truth tables **do** let us check if two propositions are equivalent.
- ✗ Truth tables **don't** give us a good way to start from a complicated proposition, and simplify it.

Strategy 2: Manipulating Expressions

Instead, we are going to learn **logical equivalence rules** to help us simplify expressions.

Similar to algebra, where we can apply rules to transform expression:

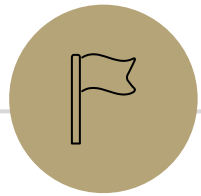
$$\underline{(x + 2)(x + 3)} = \underline{x^2 + 2x + 3x + 6}$$

Distributivity

$$= \underline{x^2 + 5x + 6}$$

Adding like terms

For each rule, we will **understand why it's true**, and **practice** using it.



Logical Equivalence Rules

Double Negation

$$\neg\neg p \equiv p$$

!!P P

I am not not loving propositional logic.

I am loving propositional logic.

De Morgan's Laws: Intuition

p : I like apples
 q : I like mangoes

Consider the following sentences:

• I don't like apples or mangoes.

$$\neg(p \vee q)$$

• I don't like apples, and I don't like mangoes.

$$\neg p \wedge \neg q$$

Are they logically equivalent? Yes?

De Morgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

De Morgan's Laws

Example: $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

De Morgan's Laws

$p: \text{front} == \text{null}$

$q: \text{value} > \text{front.data}$

```
if (!(front != null && value > front.data)) {...}
```

$$\neg (\neg p \wedge q)$$

$$\equiv \neg \neg p \vee \neg q$$

$$\equiv p \vee \neg q$$

De Morgan's

Double Negation

```
if (front == null || value <= front.data)
```

Law of Implication

Implications are unusual. Can we write them using ANDs ORs & NOTs?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \wedge \neg q$	$\neg(p \wedge \neg q)$
F	T
T	F
F	T
F	T

$$\begin{aligned}\neg(p \wedge \neg q) &\equiv \neg p \vee \neg\neg q && \text{De Morgan's} \\ &\equiv \neg p \vee q && \text{Double} \\ &&& \text{Negation}\end{aligned}$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Law of Implication: Intuition

$$\underline{p \rightarrow q \equiv \neg p \vee q}$$

If it is raining, then I have my umbrella.

$$p \rightarrow q$$

rain \checkmark umbrella \times

It is not raining, or I have my umbrella.

$$\neg p \vee q$$

\checkmark rain \checkmark umbrella \times

Converse & Contrapositive

Implication: $p \rightarrow q$

If it's raining, I have my umbrella.

Converse: $q \rightarrow p$

If I have my umbrella, it's raining.

Contrapositive: $\neg q \rightarrow \neg p$

If I don't have my umbrella, it's not raining.

Converse & Contrapositive

Implication: $p \rightarrow q$

Converse: $q \rightarrow p$

Contrapositive: $\neg q \rightarrow \neg p$

How do these relate?

implication
✓
converse

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	F	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Contrapositive: Intuition

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

If an animal is a cat, then it is a mammal.

If an animal is not a mammal, then it's not a cat.



Commutativity

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

It is raining or it is June.

It is June or it is raining.

Associativity

$$(p \vee q) \vee (r \vee s) \\ \equiv r \vee (q \vee s) \vee p$$

$$(p \vee q) \vee r \equiv p \vee (q \vee r) \\ p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$$

They perform at 3:00 and 5:00, and also 8:00.

$$(p \wedge q) \wedge r$$

They perform at 3:00, and also 5:00 and 8:00.

$$\cancel{p \wedge (q \vee r)} \\ \equiv \cancel{(p \wedge q) \vee r}$$

WARNING

Only apply associativity when all connectives are AND, or all connectives are OR

Exercise

$$A \rightarrow B \equiv \neg A \vee B$$

Prove that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ using the logical equivalences rules we've discussed so far.

Do not use contrapositive in the proof.

$$\begin{aligned} \underbrace{\neg q} \rightarrow \underbrace{\neg p} &\equiv \neg \neg q \vee \neg p && \underline{\text{Law of Imp.}} \\ &\equiv q \vee \neg p && \underline{\text{Double Neg.}} \\ &\equiv \neg p \vee q && \underline{\text{Commut.}} \\ &\equiv p \rightarrow q && \underline{\text{Law of Imp.}} \end{aligned}$$

Distributivity

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Distributivity: Intuition

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

You go to class, and you read the notes or the textbook.

You go to class and read the notes, or you go to class and you read the textbook.

Identity

$$p \wedge T \equiv p$$

$$p \vee F \equiv p$$

p	$p \wedge T$	$p \vee F$
T	T	T
F	F	F

Domination

$$p \vee T \equiv T$$

$$p \wedge F \equiv F$$

p	$p \vee T$	$p \wedge F$
T	T	F
F	T	F

Idempotency

$$p \vee p \equiv p$$

$$p \wedge p \equiv p$$

p	$p \vee p$	$p \wedge p$
T	T	T
F	F	F

Negation

$$p \vee \neg p \equiv \text{T}$$

$$p \wedge \neg p \equiv \text{F}$$

p	$p \vee \neg p$	$p \wedge \neg p$
T	T	F
F	T	F

Negation Intuition

$$p \vee \neg p \equiv \text{T}$$

$$p \wedge \neg p \equiv \text{F}$$

It is raining or it is not raining.

Always true

It is raining and it is not raining.

Always false

Absorption

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

Exercise: Build the truth tables to confirm.

Absorption

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T	T				
T	F				
F	T				
F	F				