CSE 311: Foundations of Computing I

Practice Midterm Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- You have **1 hour** to complete the exam.
- There are 5 problems on this exam, totaling 100 points.
- The exam is closed book. You may not use cell phones or calculators. You may only use the reference sheets provided.
- All answers you want graded should written on the exam paper.
- If you need extra space, use the back of a page.

1. Predicate Translation [15 points]

Let the domain of discourse be novels, comic books, movies, and TV shows. Translate the following statements to predicate logic, using the following predicates:

Novel(x) := x is a novel
Comic(x) := x is a comic book
Movie(x) := x is a movie
Show(x) := x is a TV show
Adaptation(x, y) := x is an adaptation of y

(a) (5 points) A novel cannot be adapted into both a movie and a TV show.

Solution:

$\forall x(\text{Novel}(x) \to \forall m \forall s((\text{Movie}(m) \land \text{Show}(s)) \to \neg(\text{Adaptation}(m, x) \land \text{Adaptation}(s, x)))$

(b) (5 points) Every movie is an adaptation of a novel or a comic book.

Solution:

$\forall m(\text{Movie}(m) \to \exists x(\text{Adaptation}(m, x) \land (\text{Novel}(x) \lor \text{Comic}(x))))$

(c) (5 points) Every novel has been adapted into exactly one movie.

Solution:

$\forall x(\text{Novel}(x) \to \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n((\text{Movie}(n) \land (n \neq m)) \to \neg\text{Adaptation}(n, x))))$

OR

$\forall x(\text{Novel}(x) \to \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n(\text{Adaptation}(n, x) \to (\neg\text{Movie}(n) \lor n = m))))$

OR

$\forall x(\text{Novel}(x) \to \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n((\text{Adaptation}(n, x) \land \text{Movie}(n)) \to (n = m))))$

*Note that a great exercise is to show that the above 3 solutions are all logically equivalent :)

2. CNF and DNF [15 points]
The boolean function f takes in three boolean inputs x_1, x_2, x_3, and outputs $\neg((x_1 \oplus x_2) \wedge x_3)$.

Note: You may write your solutions using boolean algebra or propositional logic notation.

(a) (5 points) Draw a truth table for f.

Solution:

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

(b) (5 points) Write a propositional logic expression for f in DNF form (ORs of ANDs).

Solution:

$(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3) \vee (x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3) \vee (\neg x_1 \wedge \neg x_2 \wedge \neg x_3)$

(c) (5 points) Write a propositional logic expression for f in CNF form (ANDs of ORs).

Solution:

$(\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$

3. Odd and Even [20 points]
Prove that for all integers k, if $k^3 + 1$ is odd, then k is even.

Solution:

We prove by contrapositive. Let k be an arbitrary integer. Suppose k is odd. We aim to show $k^3 + 1$ is even.

By definition of odd, $k = 2j + 1$ for some integer j. Consider $k^3 + 1$:

$$\begin{aligned} k^3 + 1 &= (2j+1)^3 + 1 \\ &= (2j+1)(4j^2 + 4j + 1) + 1 \\ &= 8j^3 + 8j^2 + 2j + 4j^2 + 4j + 1 + 1 \\ &= 8j^3 + 12j^2 + 6j + 2 \\ &= 2(4j^3 + 6j^2 + 3j + 1) \end{aligned}$$

Since j is an integer, $4j^3 + 6j^2 + 3j + 1$ is an integer. Then by definition of even, $k^3 + 1$ is even.

Therefore the contrapositive statement also holds: if $k^3 + 1$ is odd, then k is even. Since k was arbitrary, the claim holds for all integers k.

4. Sets [30 points]

(a) (5 points) Compute $\mathcal{P}(\{1,2\} \times \{3\})$.

Solution:

$\{\varnothing, \{(1,3)\}, \{(2,3)\}, \{(1,3),(2,3)\}\}$

(b) (5 points) Compute $\{1,2\} \cap (\{2,3\} \times \varnothing)$.

Solution:

$\varnothing$

(c) (20 points) Prove that for all sets A, B, C that if $A \subseteq B$ and $B \subseteq C$, then $B \times A \subseteq C \times C$.

Solution:

Let A, B, C be arbitrary sets. Suppose that $A \subseteq B$ and $B \subseteq C$. Let $x \in B \times A$ be arbitrary. Then by definition of Cartesian Product, $x = (y, z)$ for $y \in B$ and $z \in A$. Then since $y \in B$ and $B \subseteq C$, $y \in C$. Similarly since $z \in A$ and $A \subseteq B$, $z \in B$. Then since $z \in B$ and $B \subseteq C$, we have $z \in C$. Therefore we have shown that $y \in C$ and $z \in C$. Then by definition of Cartesian Product, $x \in C \times C$. Since x was arbitrary, we have shown $B \times A \subseteq C \times C$.

5. Induction [20 points]
Prove by induction that $(1+\pi)^n > 1 + n\pi$ for all integers $n \geq 2$.

Solution:

1. Let P(n) be the statement "$(1+\pi)^n > 1+n\pi$". We prove P(n) for all integers $n \geq 2$ by induction.

2. Base Case: When $n = 2$, the LHS is $(1+\pi)^2 = 1 + 2\pi + \pi^2$. The RHS is $1 + 2\pi$. Since $\pi^2 > 0$, $1 + 2\pi + \pi^2 > 1 + 2\pi$, so the Base Case holds.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 2$. Then $(1+\pi)^k > 1 + k\pi$.

4. Inductive Step:

Goal: Show $P(k+1)$, i.e. show $(1+\pi)^{k+1} > 1 + (k+1)\pi$

$$
\begin{aligned}
(1+\pi)^{k+1} &= (1+\pi)(1+\pi)^k && \text{Definition of Exponent} \\
&> (1+\pi)(1+k\pi) && \text{By IH} \\
&= 1 + \pi + k\pi + k\pi^2 && \text{Algebra} \\
&= 1 + (k+1)\pi + k\pi^2 && \text{Algebra} \\
&> 1 + (k+1)\pi && \text{Since } k\pi^2 > 0
\end{aligned}
$$

Thus $(1+\pi)^{k+1} > 1 + (k+1)\pi$. So P($k+1$) holds.

5. Thus we have proven P(n) for all integers $n \geq 2$ by induction.