CSE 311: Foundations of Computing I

Practice Final Part 2 Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- You have **1 hour** to complete the exam.
- There are 3 problems on this exam, totaling 65 points.
- The exam is closed book. You may not use cell phones or calculators. You may only use the reference sheets provided.
- All answers you want graded should written on the exam paper.
- If you need extra space, use the back of a page.

1. True or False [15 points]
For the following questions, determine whether the statement is true or false. Then provide 1-3 sentences of explanation. Your explanations **do not** need to be full or formal proofs.

(a) (5 points) Any subset of a regular language is also regular.

Solution:

False. For example, we saw in class that the set $\{0^n 1^n : n \geq 0\}$ is irregular. But this is a subset of the language of all binary strings, which is regular.

(b) (5 points) There exists a language L such that there is some CFG that can generate L, but no NFA can recognize L.

Solution:

True. We showed in class that the language $L = \{0^n 1^n : n \geq 0\}$ is irregular. Then there is no NFA that can recognize L. We also saw a construction of a CFG that generates this language, so there is a CFG that can generate L.

(c) (5 points) The following NFAs recognize the same language.

Solution:

True. Both NFAs recognize the language of binary strings that end in a 0.

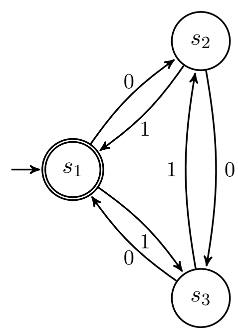
2. Models of Computation [30 points]
Consider the language of binary strings where the number of 0's minus the number of 1's is divisible by 3.

(a) (10 points) Construct a Regular Expression for this language.

Solution:
$(((0 \cup (11))(01)^*((00) \cup 1)) \cup (10))^*$

(b) (10 points) Construct a DFA for this language.

Solution:



(c) (10 points) Construct a CFG for this language.

Solution:
S $\rightarrow 0$**A** $\mid 1$**B** $\mid \varepsilon$
A $\rightarrow 0$**B** $\mid 1$**S**
B $\rightarrow 0$**S** $\mid 1$**A**

3. Structural Induction [20 points]
Let the alphabet be $\Sigma = \{0, 1, 2, ..., 9\}$. Consider the following recursive definition of the set S of strings over Σ:

Basis Step: $\varepsilon \in S$
Recursive Step: If $w \in S$ and $a \in \Sigma$ then $wa \in S$

Consider the following recursively-defined functions `double` and `sum`. Here the `int` function simply accepts a character in Σ and returns the integer value of that character.

$\texttt{double}(\varepsilon) = \varepsilon$
$\texttt{double}(wa) = \texttt{double}(w)aa$ for $w \in S, a \in \Sigma$

$\texttt{sum}(\varepsilon) = 0$
$\texttt{sum}(wa) = \texttt{sum}(w) + \texttt{int}(a)$ for $w \in S, a \in \Sigma$

Prove by structural induction that $\texttt{sum}(\texttt{double}(w)) \equiv_2 0$ for all $w \in S$.

Solution:
1. Let P(s) be "$\texttt{sum}(\texttt{double}(s)) \equiv_2 0$". We prove P($s$) for all $s \in S$ by structural induction.

2. **Base Case:** Observe that $\texttt{sum}(\texttt{double}(\varepsilon)) = \texttt{sum}(\epsilon) = 0$. Since $0 \equiv_2 0$, the base case holds.

3. **Inductive Hypothesis:** Suppose that P(w) holds for some arbitrary $w \in S$. That is, $\texttt{sum}(\texttt{double}(w)) \equiv_2 0$. Then by definition of congruence, $2 \mid \texttt{sum}(\texttt{double}(w))$. Then by definition of divides, $\texttt{sum}(\texttt{double}(w)) = 2k$ for some integer k

4. **Inductive Step:** Let $a \in \Sigma$ be arbitrary. Consider $\texttt{sum}(\texttt{double}(wa))$:

$$\begin{aligned}
\texttt{sum}(\texttt{double}(wa)) &= \texttt{sum}(\texttt{double}(w)aa) && \text{By definition of double} \\
&= \texttt{sum}(\texttt{double}(w)a) + \texttt{int}(a) && \text{By definition of sum} \\
&= \texttt{sum}(\texttt{double}(w)) + \texttt{int}(a) + \texttt{int}(a) && \text{By definition of sum} \\
&= \texttt{sum}(\texttt{double}(w)) + 2 \cdot \texttt{int}(a) && \text{Algebra} \\
&= 2k + 2 \cdot \texttt{int}(a) && \text{By IH} \\
&= 2(k + \texttt{int}(a)) && \text{Algebra}
\end{aligned}$$

Then since k and $\texttt{int}(a)$ are integers, $k+\texttt{int}(a)$ is an integer. Then by definition of divides, $2 \mid \texttt{sum}(\texttt{double}(wa))$. Then by definition of congruence, $\texttt{sum}(\texttt{double}(wa)) \equiv_2 0$. So P($wa$) holds.

5. **Conclusion:** Thus P(s) holds for all strings $s \in S$ by structural induction.