CSE 311: Foundations of Computing I

Practice Final Part 1 Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- You have **1 hour** to complete the exam.
- There are 3 problems on this exam, totaling 70 points.
- The exam is closed book. You may not use cell phones or calculators. You may only use the reference sheets provided.
- All answers you want graded should written on the exam paper.
- If you need extra space, use the back of a page.

1. True or False [30 points]

For the following questions, determine whether the statement is true or false. Then provide 1-3 sentences of explanation. Your explanations **do not** need to be full or formal proofs.

(a) (5 points) "p only if q" and "q is necessary for p", are both best translated as $p \to q$.

Solution:

True. Both translations say that in order for p to occur, q must occur. So when p happens q must have happened too. Thus $p \to q$.

(b) (5 points) One way to prove that $p \to q$ is true is to show that the converse, $q \to p$, is false.

Solution:

False. There are implications where both the converse and the original statement are true, so this is not a valid proof technique.

(c) (5 points) The implication $\forall y \exists x$ P$(x, y) \to \exists x \forall y$ P(x, y) is true regardless of what the predicate P is.

Solution:

False. In the statement $\forall y \exists x$ P(x, y), it may be that x depends on the value of y. For example, "every person has an ancestor". In the statement $\exists x \forall y$ P(x, y), it must be that the single x satisfy P for all y. For example, "there is one person that is every person's ancestor." So, the first statement does not always imply the second.

(d) (5 points) Suppose a, b, m, n are all integers greater than 1. If $a \equiv_m b$ and $m \mid n$, then $a \equiv_n b$.

Solution:

False. For example, consider $a = 3$, $b = 6$, $m = 3$, and $n = 9$. Then $a \equiv_m b$ and $m \mid n$ but $a \not\equiv_n b$

(e) (5 points) Suppose A, B are sets. If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

Solution:

True. Let $x \in A$ be arbitrary. Consider $\{x\}$, i.e. the singleton set containing just the element x. Since $x \in A$, by definition of subset $\{x\} \subseteq A$. Then by definition of powerset, $\{x\} \in \mathcal{P}(A)$. Then because $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have $\{x\} \in \mathcal{P}(B)$. Then by definition of powerset, $\{x\} \subseteq B$. Then by definition of subset, every element of $\{x\}$ is an element of B. In particular, $x \in B$. Since x was arbitrary, $A \subseteq B$.

Note: This level of detail and formalism is not required.

(f) (5 points) Strong induction proofs always require more than one base case.

Solution:

False. Not necessarily. For example, the first strong induction proof we did in class (every positive integer greater than 1 can be written as a product of primes) only required one base case.

2. Contradiction [20 points]

Recall that we defined the <u>rational numbers</u> as the set $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$. For example, $\frac{5}{8}, -12$ are rational.

We defined the <u>irrational numbers</u> as all real numbers that are not rational. For example, $\sqrt{2}, \pi$ are irrational.

Prove by contradiction that for all real numbers x, y if x is rational and xy is irrational, then y is irrational.

Solution:

Suppose for the sake of contradiction that there exist real numbers x, y such that x is rational and xy is irrational, but y is rational. Then by definition of rational, $x = \frac{a}{b}$ for some integers a, b such that $b \neq 0$, and $y = \frac{c}{d}$ for some integers c, d such that $d \neq 0$. Then observe that $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ab}{cd}$. Since a, b are integers, ab is an integer. Similarly since c, d are non-zero integers, cd is a non-zero integer. Then by definition, xy is rational. This contradicts the assumption that xy was irrational. So the claim holds that for all real numbers x, y if x is rational and xy is irrational, then y is irrational.

3. Induction [20 points]
Consider the following recursive definition of a function f defined over all $n \in \mathbb{N}$:

$$
\begin{aligned}
f(n) &= 1 && \text{if } n = 0 \\
f(n) &= 3 \cdot f(n-1) && \text{if } 1 \le n \le 2 \\
f(n) &= 3 \cdot f(n-1) - 2 \cdot f(n-2) + 6 \cdot f(n-3) && \text{for all } n > 2
\end{aligned}
$$

Prove using strong induction that, for all $n \in \mathbb{N}$, we have $f(n) = 3^n$.

Solution:

Let $P(n)$ be the claim that $f(n) = 3^n$. We will prove $P(n)$ for all $n \in \mathbb{N}$ by strong induction.

Base Case ($n = 0, 1, 2$). By definition, $f(0) = 1 = 3^0$, so $P(0)$ holds; $f(1) = 3 \cdot 1 = 3^1$, so $P(1)$ holds; and $f(2) = 3 \cdot 3 = 3^2$, so $P(2)$ holds.

Inductive Hypothesis. Suppose that $P(0) \wedge \ldots \wedge P(k)$ hold for an arbitrary $k \ge 2$.

Inductive Step. We will show that $P(k+1)$ is true. Observe:

$$
\begin{aligned}
f(k+1) &= 3 \cdot f(k) - 2 \cdot f(k-1) + 6 \cdot f(k-2) && \text{Def of } f\text{, since } k+1 \ge 3 > 2 \\
&= 3 \cdot 3^k - 2 \cdot 3^{k-1} + 6 \cdot 3^{k-2} && \text{Inductive Hypothesis (note: } k-2 \ge 0) \\
&= 3^{k-1}(3^2 - 2 + 2) && \text{Algebra} \\
&= 3^{k-1} \cdot 3^2 && \text{Algebra} \\
&= 3^{k+1} && \text{Algebra}
\end{aligned}
$$

Thus $P(k+1)$ holds.

Conclusion. Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by strong induction.