CSE 311: Foundations of Computing I

Homework 6 (due Wednesday, August 9th at 11:59 PM)

Directions: *Write up carefully argued solutions to the following problems. Each solution should be clear enough that it can explain why it works to someone who does not already understand the answer. If you work with others, remember to follow the collaboration policy outlined in the syllabus. Be sure to read the Typesetting and Grading guidelines prior to submitting.*

1. I Don't Always Contradict Myself (But When I Do, I Don't) (24 points)

Prove the following claims by contradiction.

(a) [12 Points] There does not exist integers a, b such that $3a - 9b = 2$.

(b) [12 Points] For all real numbers $x > 0$, $x + \frac{1}{x} \geq 2$.

2. Barking Up the Strong Tree (18 points)

The local grocery store sells candies in packs of 3 and 11.

(a) [2 Points] Find the smallest integer value of x such that for all integers $n \geq x$, you can buy n candies.

(b) [16 Points] Now prove your answer in part (a) by strong induction.

3. Am I Strong Enough? (16 points)

The sequence g_n is defined recursively for integers $n \geq 1$ as follows:

$$g_1 = 1$$
$$g_2 = 4$$
$$g_3 = 9$$
$$g_4 = 16$$
$$g_n = g_{n-1} - g_{n-2} + g_{n-3} + 2(2n-3) \qquad \text{for all } n \geq 5$$

Prove that for all integers $n \geq 1$ that $g_n = n^2$.

4. Recursively Defined Sets (12 points)

For each of the following, write a recursive definition of the set of strings satisfying the given properties. Then *briefly* justify why your solution is correct. You **do not** need to write a formal proof.

(a) [4 Points] Binary strings where every 0 is immediately followed by a 1.

(b) [4 Points] Binary strings that start with 0 and have even length.

(c) [4 Points] Binary strings with an even number of 0s.

5. Treeshake (16 points)

We define simple binary trees as the recursive set $\mathcal{B}$:
Basis Step: $\bullet \in \mathcal{B}$.
Recursive Step: If $L, R \in \mathcal{B}$, then $(L, \bullet, R) \in \mathcal{B}$.

Note that these are slightly different than the trees defined in class. These trees cannot be null.

Define the following functions on simple binary trees:

$$\texttt{edges}(t) = \begin{cases} 0 & \text{if } t = \bullet \\ 2 + \texttt{edges}(L) + \texttt{edges}(R) & \text{if } t = (L, \bullet, R) \end{cases}$$

$$\texttt{degree}(t) = \begin{cases} 1 & \text{if } t = \bullet \\ 3 & \text{if } t = (L, \bullet, R) \end{cases}$$

$$\texttt{sum}(t) = \begin{cases} \texttt{degree}(t) & \text{if } t = \bullet \\ \texttt{degree}(t) + \texttt{sum}(L) + \texttt{sum}(R) & \text{if } t = (L, \bullet, R) \end{cases}$$

Prove that for all $t \in \mathcal{B}$, $\texttt{sum}(t) = 2 \cdot \texttt{edges}(t) + 1$.

This is a special case of an incredibly important result in graph theory called the *Handshaking Lemma*. You will probably use it a lot if you end up taking CSE 421 :)

6. Feedback (2 points)

Please share approximately how many hours you spent working on the **non-optional portion** of this assignment (see optional portion below!). Report your estimate to the nearest hour. This will help us calibrate our assignments in the future.

If you have any additional feedback, we welcome that as well.

7. Optional: That's just Plane Nonsense! (0 points)

The goal of this problem is to develop a way of visualizing rational numbers. We will work with the upper right quadrant of the 2D plane, specifically we are interested in the points which have strictly integer coordinates, which we will call integer points. For example, $(0, 1)$ is an integer point because 0 and 1 are integers, but $(0.5, \pi)$ is not. Imagine drawing a line with slope m from the origin, $y = mx$. Below is a diagram showing two examples.

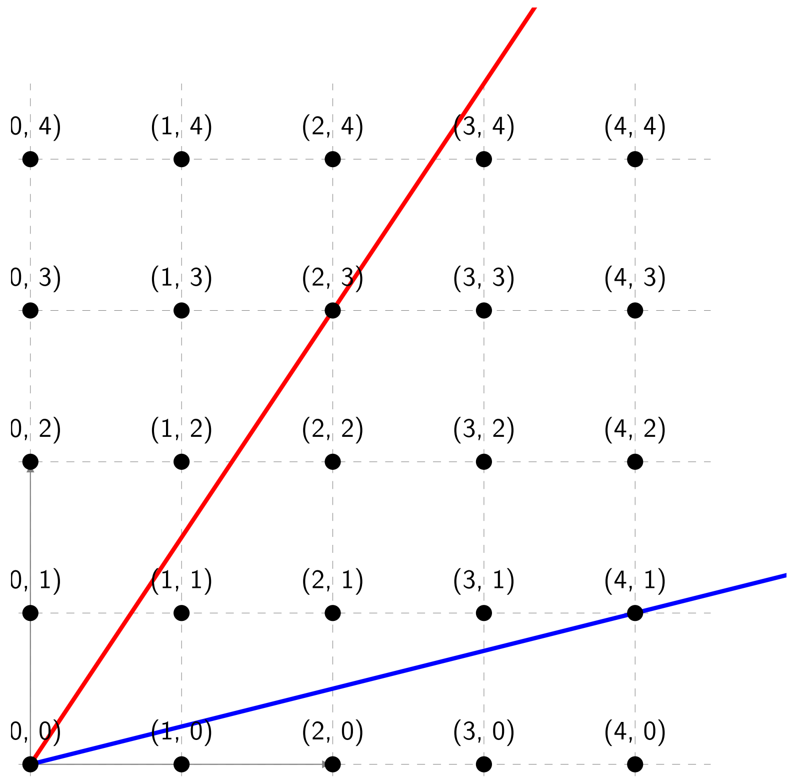

Figure 1: $\frac{2}{3}x$ and $\frac{1}{4}x$ graphed on the upper right quadrant of the 2D plane

(a) [0 Points] Prove that if m is irrational if and only if the line $y = mx$ does not intersect any integer points. Remember an if and only if is asking you for two proofs, one in each direction.
Hint: You might find it easier if you use contradiction in both directions.

Imagine standing at the origin (the bottom left corner) and looking up to the right at all the integer points. Notice that some points are "in the way" of each other, for instance, you cannot see $(2, 2)$ because it is behind $(1, 1)$, but you can see $(3, 1)$, $(4, 1)$, and even $(1000, 1)$ (assume you have very good vision). Formally, we say that an integer point is **unobscured** if and only if you can draw a straight line from the origin to the point without intersecting any other integer point. Below is a diagram illustrating some of the unobscured integer points which are close to the origin. But notice that there are actually infinitely many unobscured integer points.

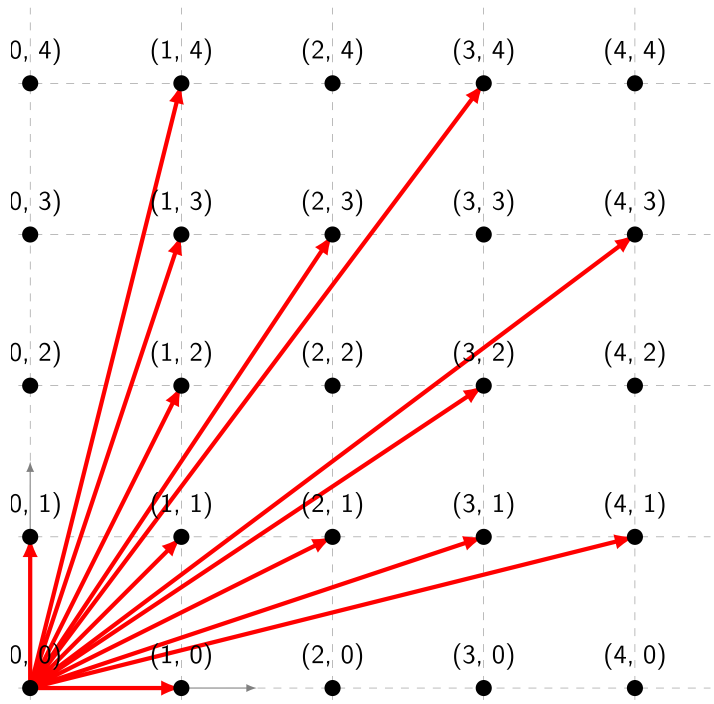

Figure 2: Arrows pointing to all the unobscured integer points which have coordinates ≤ 4.

To put our visual definition of an unobscured point into formal logic, we say that a point (a, b) is **unobscured** if and only if

$$\neg\exists c\exists d[(c < a) \wedge (d < b) \wedge (a/b = c/d)]$$

where a, b, c, d are positive integers.

Take a moment to think about how this definition is equivalent to the visual explanation. (Remember, the line connecting the origin and (a, b) is $y = (a/b)x$, and the line connecting the origin and (c, d) is $y = (c/d)x$, so if $a/b = c/d$ then in fact they are on the same line!)

(b) [0 Points] Recall that a fraction a/b is called "reducible" if it can be simplified by dividing both sides by some number. For instance, $4/16$ can be reduced to $1/4$ by dividing both sides by 4, but $1/4$ is not reducible. In other words, the greatest common divisor of a and b is 1. Formally, a/b is reducible if

$$\exists x(x > 1 \wedge x|a \wedge x|b).$$

That is, there exists an integer greater than 1 which divides both a and b.

Prove that if fraction a/b is reducible then the point (a, b) is **not** an **unobscured** integer point.
Hint: Use the fact that a/b is reducible to find another integer point on the line which is closer to the origin.

(c) [0 Points] Challenge: The last two parts are very hard and require probability, which we have not learned in this class.

Using part (b), prove that if you chose an integer point from the 2D plane uniformly at random, then the probability that it is unobscured is $\prod_p (1 - 1/p^2)$ where $\prod_p$ is the product taken over all primes p. You may assume without proof that a random integer is divisible by a prime p with probability $1/p$.

(d) [0 Points] Challenge: Use the previous part, and the following facts[1]

$$\prod_p \frac{1}{1 - 1/p^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

to conclude that the probability of a randomly chosen integer point being unobscured is $6/\pi^2$.

[1] If you are interested, they are proved here: 3blue1brown's video proof of the Basel Problem, Euler's Product Formula.