Quiz Section 10: Cardinality and Computability – Solutions

Task 1 – Cardinality

a) You are a pirate. You begin in a square on a 2D grid that is infinite in all directions. In other words, wherever you are, you may move up, down, left, or right. Some single square on the infinite grid has treasure on it. Find a way to ensure you find the treasure in finitely many moves.

Explore the square you are currently on. Explore the unexplored perimeter of the explored region until you find the treasure (your path will look a bit like a spiral).

b) Prove that $\{3x : x \in \mathbb{N}\}$ is countable.

We can enumerate the set as follows:

$$
\begin{aligned}
f(0) &= 0 \\
f(1) &= 3 \\
f(2) &= 6 \\
f(i) &= 3i
\end{aligned}
$$

Since every natural number appears on the left, and every number in S appears on the right, this enumeration spans both sets, so S is countable.

c) Prove that the set of irrational numbers is uncountable.
Hint: Use the fact that the rationals are countable and that the reals are uncountable.

We first prove that the union of two countable sets is countable. Consider two arbitrary countable sets C_1 and C_2. We can enumerate $C_1 \cup C_2$ by mapping even natural numbers to C_1 and odd natural numbers to C_2.
Now, assume that the set of irrationals is countable. Then the reals would be countable, since the reals are the union of the irrationals (countable by assumption) and the rationals (countable). However, we have already shown that the reals are uncountable, which is a contradiction. Therefore, our assumption that the set of irrationals is countable is false, and the irrationals must be uncountable.

d) Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

Assume for the sake of contradiction that $\mathcal{P}(\mathbb{N})$ is countable.
This means we can define an enumeration of elements S_i in $\mathcal{P}$.
Let s_i be the binary set representation of S_i in $\mathbb{N}$. For example, for the set $\{0, 1, 2\}$, the binary set representation would be $111000\ldots$
We then construct a new subset $X \subset \mathbb{N}$ such that $x[i] = \sim s_i[i]$ (that is, $x[i]$ is 1 if $s_i[i]$ is 0, and $x[i]$ is 0 otherwise).
Note that X is not any of S_i, since it differs from S_i on the ith natural number. However, X still represents a valid subset of the natural numbers, which means our enumeration is incomplete, which is a contradiction. Since the above proof works for any listing of $\mathcal{P}(\mathbb{N})$, *no* listing can be created for $\mathcal{P}(\mathbb{N})$, and therefore $\mathcal{P}(\mathbb{N})$ is uncountable.

Task 2 – Countable Unions

a) Show that $\mathbb{N} \times \mathbb{N}$ is countable.

Hint: How did we show that the rationals were countable?

We use dovetailing to create a sequence of elements of $\mathbb{N} \times \mathbb{N}$ that includes the entirety of $\mathbb{N} \times \mathbb{N}$.
For a fixed integer $k \geqslant 2$, consider subset S_k of $\mathbb{N} \times \mathbb{N}$ consisting of the elements (a, b) such that $a + b = k$. There can be at most $k - 1$ such elements because for each value of $a = 1, 2, \ldots, k - 1$, there can only be one possible value for b, namely $k - a$. Thus, if we create a sequence consisting of all the elements of S_2, then S_3, then S_4, etc. because each set is of finite size, any pair $(a, b) \in \mathbb{N} \times \mathbb{N}$ will eventually show up in this sequence in S_{a+b}.
Thus, because we can enumerate the elements of $\mathbb{N} \times \mathbb{N}$, it must be countable.

b) Show that the countable union of countable sets is countable. That is, given a collection of sets $S_1, S_2, S_2, \ldots$ such that S_i is countable for all $i \in \mathbb{N}$, show that

$$S = S_1 \cup S_2 \cup \cdots = \{x : x \in S_i \text{ for some } i\}$$

is countable.

Hint: Find a way of labeling the elements and see if you can apply the previous part to construct an onto function from $\mathbb{N}$ to S.

Because each S_i is countable, the elements can be enumerated. Let the elements of S_i be $a_{i,1}, a_{i,2}, a_{i,3}, \ldots$. Next, because $\mathbb{N} \times \mathbb{N}$ is countable, there exists an onto function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Then define the function $g : \mathbb{N} \to S$ as follows. For each $n \in \mathbb{N}$, let $(i_n, j_n) = f(n)$. The define $g(n)$ to be a_{i_n, j_n}.
I claim g is onto. Indeed, let $a_{i,j}$ be an arbitrary element of S. Because f is onto, there exists an n such that $f(n) = (i, j)$. Then $g(n) = a_{i,j}$. This shows g is onto and thus S is countable.

Task 3 – Computability

Let $\Sigma = \{0,1\}$. Prove that the set of palindromes over alphabet Σ is decidable.

We can implement the function that takes a string as input and reverses that string, using the recursive definition of string reverse given in class. So on input x we run that reversing program to create the string $y = x^R$. Then we compare x against y character by character and output yes iff we find that $x = y$.

Task 4 – Review: Strong Induction

Define a sequence of positive integers a_n with $n \geqslant 1$ as follows:

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= 2 \\
a_3 &= 5 \\
a_n &= 3a_{n-1} + 4a_{n-2} + a_{n-3} && \text{for } n \geqslant 4
\end{aligned}$$

Prove that $a_n \geqslant 4^{n-2}$ for all integers $n \geqslant 1$.

We will proceed by strong induction.

Let $P(n)$ be "$a_n \geqslant 4^{n-2}$" . We will prove that $P(n)$ holds for all integers $n \geqslant 1$.

Base Cases. When $n = 1$, by definition of a, we have $a_n = a_1 = 1$. Also, $4^{n-2} = 4^{-1} = 1/4$. Since $1 \geqslant 1/4$ we have $P(1)$ holds. When $n = 2$, by definition of a, we have $a_n = a_2 = 2$. Also, $4^{n-2} = 4^{2-2} = 4^0 = 1$. Since $2 \geqslant 1$, it follows that $P(2)$ holds. When $n = 3$, by definition of a, we have $a_n = a_3 = 5$. Also, $4^{n-2} = 4^{3-2} = 4^1 = 4$. Since $5 \geqslant 4$, it follows that $P(3)$ holds.

Inductive Hypothesis. Let k be some integer with $k \geqslant 3$, and and suppose that $P(j)$ holds for all j such that $1 \leqslant j \leqslant k$.

Inductive Step. $\boxed{\text{Goal: Show } P(k+1)\text{. Or, simplifying, show } a_{k+1} \geqslant 4^{k-1}.}$

Since $k \geqslant 3$, we have $k + 1 \geqslant 4$.

We then calculate:

$$\begin{aligned}
a_{k+1} &= 3a_k + 4a_{k-1} + a_{k-2} && \text{by def of } a\text{, since } k+1 \geqslant 4 \\
&\geqslant 3 \cdot 4^{k-2} + 4 \cdot 4^{k-3} + 4^{k-4} && \text{by IH with } j = k,\ k-1,\ k-2 \geqslant 1 \text{ (since } k \geqslant 3\text{)} \\
&= 3 \cdot 4^{k-2} + 4^{k-2} + 4^{k-4} \\
&= 4 \cdot 4^{k-2} + 4^{k-4} \\
&= 4^{k-1} + 4^{k-4} \\
&\geqslant 4^{k-1}
\end{aligned}$$

Thus $P(k+1)$ holds.

Conclusion. By strong induction, $P(n)$ holds for all integers $n \geqslant 1$.