Quiz Section 3: Predicate Logic and Inference – Solutions

Review

Inference Rules

$\boxed{\text{Excluded Middle}}\quad \dfrac{}{\therefore\ A \vee \neg A}$

$\boxed{\text{Intro } \wedge}\quad \dfrac{A;\ B}{\therefore\ A \wedge B}$

$\boxed{\text{Elim } \wedge}\quad \dfrac{A \wedge B}{\therefore\ A,\ B}$

$\boxed{\text{Intro } \vee}\quad \dfrac{A,\ B}{\therefore\ A \vee B}$

$\boxed{\text{Elim } \vee}\quad \dfrac{A \vee B;\ \neg A}{\therefore\ B}$

$\boxed{\text{Direct Proof}}\quad \dfrac{A \Rightarrow B}{\therefore\ A \rightarrow B}$

$\boxed{\text{Modus Ponens}}\quad \dfrac{A;\ A \rightarrow B}{\therefore\ B}$

$\boxed{\text{Intro } \exists}\quad \dfrac{P(c) \text{ for some } c}{\therefore\ \exists x\ P(x)}$

$\boxed{\text{Elim } \forall}\quad \dfrac{\forall x\ P(x)}{\therefore\ P(a) \text{ for any } a}$

$\boxed{\text{Intro } \forall}\quad \dfrac{\text{Let } a \text{ be arbitrary} \ldots P(a)}{\therefore\ \forall x\ P(x) \qquad \text{(If no other name in } P \text{ depends on } a\text{)}}$

$\boxed{\text{Elim } \exists}\quad \dfrac{\exists x\ P(x)}{\therefore\ P(c) \text{ for some } \textit{special } c \qquad \text{list dependencies for } c}$

Task 1 – Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other, if they are equivalent, or neither.

a) $\forall x\ \forall y\ P(x, y)$ $\qquad$ $\forall y\ \forall x\ P(x, y)$

These sentences are the same; switching universal quantifiers makes no difference.

b) $\exists x\ \exists y\ P(x, y)$ $\qquad$ $\exists y\ \exists x\ P(x, y)$

These sentences are the same; switching existential quantifiers makes no difference.

c) $\forall x\ \exists y\ P(x, y)$ $\qquad$ $\forall y\ \exists x\ P(x, y)$

These are only the same if P is symmetric (i.e., the order of the arguments doesn't matter). If the order of the arguments does matter, then these are different statements. For instance, if $P(x, y)$ is "$x < y$", then the first statement says "for every x, there is a corresponding y such that $x < y$", whereas the second says "for every y, there is a corresponding x such that $x < y$". In other words, in the first statement y is a function of x, and in the second x is a function of y.

If your domain of discourse is "positive integers", for example, the first is true and the second is false; but for "negative integers" the second is true while the first is false.

d) $\forall x\, \exists y\, P(x, y)$ $\qquad\qquad$ $\exists x\, \forall y\, P(x, y)$

These two statements are usually different.

e) $\forall x\, \exists y\, P(x, y)$ $\qquad\qquad$ $\exists y\, \forall x\, P(x, y)$

The second statement is "stronger" than the first (that is, the second implies the first). For the first, y is allowed to depend on x. For the second, one specific y must work for all x. Thus if the second is true, whatever value of y makes it true, can also be plugged in for y in the first statement for every x. On the other hand, if the first statement is true, it might be that different y's work for the different x's and no single value of y exists to make the latter true.

As an example, let you domain of discourse be positive real numbers, and let $P(x, y)$ be $xy = 1$. The first statement is true (always take y to be $1/x$, which is another positive real number). The second statement is not true; it asks for a single number that always makes the product 1.

Task 2 – Quantifier Ordering

Let your domain of discourse be a set of `Element` objects given in a list called `Domain`. Imagine you have a predicate $\texttt{pred}(x, y)$, which is encoded in the java method `public boolean pred(int x, int y)`. That is you call your predicate `pred` true if and only if the java method returns `true`.

a) Consider the following Java method:

```
public boolean Mystery(Domain D){
    for(Element x : D) {
        for(Element y : D) {
            if(pred(x,y))
                return true;
        }
    }
}
```

`Mystery` corresponds to a quantified formula (for `D` being the domain of discourse), what is that formula?

$\exists x \exists y(\texttt{pred}(x, y))$. If any combination of x and y causes pred to evaluate to true, we return true; that is we just want x, y to exist.

b) What formula does `mystery2` correspond to

```
public boolean Mystery2(Domain D){
    for(Element x : D) {
        boolean thisXPass = false;
        for(Element y : D) {
            if(pred(x,y))
                thisXPass = true;
        }
        if(!thisXPass)
            return false;
    }
    return true;
}
```

$\forall x \exists y(\texttt{pred}(x, y))$.

For a given x, when we come across a y that makes $\texttt{pred}(x, y)$ true, we set the given x to pass (so one y suffices for a given x) but we require **every** x to pass, so x is universally quantified. Since y is allowed to depend on x, we have x as the outermost variable.

Task 3 – Find the Bug

Each of these inference proofs is incorrect. Identify the line (or lines) which incorrectly apply a law, and explain the error. Then, if the claim is false, give concrete examples of propositions to show it is false. If it is true, write a correct proof.

a) This proof claims to show that given $a \to (b \vee c)$, we can conclude $a \to c$.

1. $a \to (b \vee c)$ [Given]

2.1.	a	[Assumption]
2.2.	$\neg b$	[Assumption]
2.3.	$b \vee c$	[Modus Ponens, from 1 and 2.1]
2.4.	c	[$\vee$ elimination, from 2.2 and 2.3]

2. $a \to c$ [Direct Proof Rule, from 2.1-2.4]

The error here is in lines 2.2 and 2. When beginning a subproof for the direct proof rule, only one assumption may be introduced. If the author of this proof wanted to assume both a and $\neg b$, they should have used the assumption $a \wedge \neg b$, which would make line 3 be $(a \wedge \neg b) \to c$ instead.

And the claim is false in general. Consider:
a: "I am outside"
b: "I am walking my dog"
c: "I am swimming"
If we assert "If I am outside, I am walking my dog or swimming," we cannot reasonably conclude that "If I am outside, I am swimming" ($a \to c$).

b) This proof claims to show that given $p \rightarrow q$ and r, we can conclude $p \rightarrow (q \vee r)$.

1. $p \rightarrow q$	[Given]
2. r	[Given]
3. $p \rightarrow (q \vee r)$	[Intro $\vee$ (1,2)]

Bug is in step 3, we're applying the rule to only a subexpression.

The statement is true though. A correct proof introduces p as an assumption, uses MP to get q, introduces $\vee$ to get $q \vee r$, and the direct proof rule to complete the argument.

c) This proof claims to show that given $p \rightarrow q$ and q that we can conclude p

1. $p \rightarrow q$	[Given]
2. q	[Given]
3. $\neg p \vee q$	[Law of Implication (1)]
4. p	[Eliminate $\vee$ (2,3)]

The bug is in step 4. Eliminate $\vee$ from 3 would let us conclude $\neg p$ if we had $\neg q$ or q if we had p. It doesn't tell us anything with knowing q.

Indeed, the claim is false. We could have p: "it is raining"
q: "I have my umbrella"

And be a person who always carries their umbella with them (even on sunny days). The information is not sufficient to conclude p.

Task 4 – Formal Proof (Direct Proof Rule)

Show that $\neg t \rightarrow s$ follows from $t \vee q$, $q \rightarrow r$ and $r \rightarrow s$ with a formal proof. Then, translate your proof to English. You can try this problem on Cozy at https://bit.ly/cse311-23sp-section03-4

Formal proof:

1.	$t \vee q$		[Given]
2.	$q \rightarrow r$		[Given]
3.	$r \rightarrow s$		[Given]
	4.1. $\neg t$	[Assumption]	
	4.2. q	[Elim of $\vee$: 1, 4.1]	
	4.3. r	[MP of 4.2, 2]	
	4.4. s	[MP 4.3, 3]	
4.	$\neg t \rightarrow s$		[Direct Proof Rule]

English proof:
Suppose $\neg t$. Since we are given that t holds or q holds, we know that q holds. Given this, and that we know q implies r, we know r holds. We also know r implies s, so s must hold.

Task 5 – Formal Proof

Show that $\neg p$ follows from $\neg(\neg r \vee t)$, $\neg q \vee \neg s$ and $(p \to q) \wedge (r \to s)$ with a formal proof. Then, translate your proof to English. You can try this problem on Cozy at https://bit.ly/cse311-23sp-section03-5.

Formal proof:

1.	$\neg(\neg r \vee t)$	[Given]
2.	$\neg q \vee \neg s$	[Given]
3.	$(p \to q) \wedge (r \to s)$	[Given]
4.	$\neg\neg r \wedge \neg t$	[DeMorgan's Law: 1]
5.	$\neg\neg r$	[Elim of $\wedge$: 4]
6.	r	[Double Negation: 5]
7.	$r \to s$	[Elim of $\wedge$: 3]
8.	s	[MP, 6,7]
9.	$\neg\neg s$	[Double Negation: 8]
10.	$\neg s \vee \neg q$	[Commutative: 2]
11.	$\neg q$	[Elim of $\vee$: 10, 9]
12.	$p \to q$	[Elim of $\wedge$: 3]
13.	$\neg q \to \neg p$	[Contrapositive: 12]
14.	$\neg p$	[MP: 11,13]

English proof:
We are given that neither $\neg r$ nor t is true, which tells us that, r (and $\neg t$) must hold. We know that r implies s, so s must also hold. Since we know that $\neg s$ holds or $\neg q$ holds, and s holds, it must be that $\neg q$ holds. We were also given that p implies q, so taking the contrapositive, we know that $\neg q$ implies $\neg p$. Since we know $\neg q$, we can conclude that $\neg p$ holds.

Task 6 – All for 1 and One $\forall$

Let the domain of discourse contain only the two object a and b. *For this problem only*, you are allowed to use the following <u>fake</u> equivalence rules

$$\forall x\, P(x) \equiv P(a) \wedge P(b) \qquad \forall \to \wedge$$
$$\exists x\, P(x) \equiv P(a) \vee P(b) \qquad \exists \to \vee$$

In this question, Q will stand for some arbitrary fully quantified predicate logic formula.

a) Use a chain of equivalences to show that $Q \wedge (\exists x\, P(x)) \equiv \exists x\, (Q \wedge P(x))$.

$$\begin{aligned} Q \wedge (\exists x\, P(x)) &\equiv Q \wedge (P(a) \vee P(b)) && \exists \to \vee \\ &\equiv (Q \wedge P(a)) \vee (Q \wedge P(b)) && \text{Distributivity} \\ &\equiv \exists x\, (Q \wedge P(x)) && \exists \to \vee \end{aligned}$$

b) Likewise show that $Q \lor (\exists x\, P(x)) \equiv \exists x\, (Q \lor P(x))$.

$$\begin{aligned}
Q \lor (\exists x\, P(x)) &\equiv Q \lor (P(a) \lor P(b)) && \exists \to \lor \\
&\equiv (Q \lor Q) \lor (P(a) \lor P(b)) && \text{Idempotence} \\
&\equiv Q \lor (Q \lor (P(a) \lor P(b))) && \text{Associativity} \\
&\equiv Q \lor ((Q \lor P(a)) \lor P(b)) && \text{Associativity} \\
&\equiv (Q \lor (Q \lor P(a)) \lor P(b) && \text{Associativity} \\
&\equiv ((Q \lor P(a)) \lor Q) \lor P(b) && \text{Commutativity} \\
&\equiv (Q \lor P(a)) \lor (Q \lor P(b)) && \text{Associativity} \\
&\equiv \exists x\, (Q \lor P(x)) && \exists \to \lor
\end{aligned}$$

c) Are each of these equivalences also true assuming our fake equivalences? Yes or no.

i $Q \land (\forall x\, P(x)) \equiv \forall x\, (Q \land P(x))$

ii $Q \lor (\forall x\, P(x)) \equiv \forall x\, (Q \lor P(x))$.

Yes. All three are easy adaptations of the proofs above.

d) Do the equivalences proven in (a)-(b) hold in every other domain of discourse? Briefly explain why or why not.

Yes. The fake equivalences can become infinitely long, but the proven equivalences remain true.

Task 7 – Proof, Goof, or Spoof?

For each of the claims below, (1) translate the English proof into a formal proof and (2) say which of the following categories describes the formal proof:

Proof The proof is correct.

Goof The claim is true but the proof is wrong.

Spoof The claim is false.

Finally, (3) if it is a goof, point out the errors in the proof and explain how to correct them, and if it is a spoof, point out the *first* error in the proof and then show that the claim is false by giving a counterexample. (If it is a correct proof, then skip part (3).)

a) Show that r follows from $\neg p$ and $p \leftrightarrow r$.

Proof, Goof, or Spoof: Since we are given that $p \leftrightarrow r$, we know $p \to r$. We are also given that $\neg p$ holds, so it must be the case that $\neg p \lor (p \lor r)$ holds. This claim is equivalent to $(p \land \neg p) \to r$. Since this last claim starts by assuming both p and $\neg p$, we can infer that this holds with just $\neg p$, giving us $\neg p \to r$. Since we were given that $\neg p$ holds, we get that r holds.

1.	$p \leftrightarrow r$	Given
2.	$(p \to r) \land (r \to p)$	Defn Biconditional: 1
3.	$p \to r$	Elim $\land$: 2
4.	$\neg p$	Given
5.	$\neg p \lor (p \lor r)$	Intro $\lor$: 4
6.	$(\neg p \lor p) \lor r$	Associativity: 5
7.	$(\neg\neg\neg p \lor \neg\neg p) \lor r$	Double Negation: 6
8.	$\neg(\neg\neg p \land \neg p) \lor r$	De Morgan's: 7
9.	$\neg(p \land \neg p) \lor r$	Double Negation: 8
10.	$(p \land \neg p) \to r$	Law of Implication: 9
11.	$\neg p \to r$	Elim $\land$: 10
12.	r	Modus Ponens: 4, 11

This is a Spoof. On line 10, Elim $\land$ is applied to a subexpression, which is not valid. The conclusion of the proof is false, which we can see by considering the following counterexample. If $p \equiv F$ and $r \equiv F$, then $\neg p \equiv T$ and $p \leftrightarrow r \equiv F \leftrightarrow F$ is true (i.e. the givens are true), but r, the conclusion, is false.

b) Show that $\exists z\ \forall x\ P(x, z)$ follows from $\forall x\ \exists y\ P(x, y)$.

Proof, Goof, or Spoof: We are given that, for every x, there is some y such that P(x, y) holds. Thus, there must be some object c such that for every x, P(x, c) holds. This shows that there exists an object z such that, for every x, P(x, z) holds.

1.	$\forall x\ \exists y\ P(x, y)$	Given
2.	$\forall x\ P(x, c)$	$\exists$ Elim: 1 (c special)
3.	$\exists z\ \forall x P(x, z)$	$\exists$ Intro: 2

This is a Spoof. The mistake is on line 2 where an inference rule is used on a subexpression. When we apply something like the $\exists$ Elim rule, the $\exists$ must be at the start of the expression and outside all other parts of the statement.

The conclusion is false, it's basically saying we can interchange the order of $\forall$ and $\exists$ quantifiers. Let the domain of discourse be integers and define $P(x, y)$ to be $x < y$. Then the hypothesis is true: for every integer, there is a larger integer. However, the conclusion is false: there is no integer that is larger than every other integer. Hence, there can be no correct proof that the conclusion follows from the hypothesis.

c) Show that $\exists z\ (P(z) \land Q(z))$ follows from $\forall x\ P(x)$ and $\exists y\ Q(y)$.

Proof, Goof, or Spoof: Let z be arbitrary. Since we were given that for every x, P(x) holds, P(z) must hold. Since we were given that there is a y such that Q(y) holds, Q(z) must also hold. From the previous facts, we know that there is some object z such that P(z) and Q(z) hold.

1.	$\forall x\ P(x)$	Given
2.	$\exists y\ Q(y)$	Given
3.	Let z be arbitrary	
4.	$P(z)$	Elim $\forall$: 1
5.	$Q(z)$	Elim $\exists$: 2 (z special)
6.	$P(z) \wedge Q(z)$	Intro $\wedge$: 4, 5
7.	$\exists z\ P(z) \wedge Q(z)$	Intro $\exists$: 6

This is a Goof. The mistake is on line 5. The $\exists$ Elim rule must create a new variable rather than applying some property to an existing variable.

The conclusion is true in this case. Instead of declaring z to be arbitrary and then applying $\exists$ Elim to make it specific, we can instead just apply the $\exists$ Elim rule directly to create z. To do this, we would remove lines 3 and 5 and define z by applying $\exists$ Elim to line 2. Note, it's important that we define z before applying line 4.

Task 8 – Predicate Logic Formal Proof

Given $\forall x\,(T(x) \to M(x))$, we wish to prove $(\exists x\, T(x)) \to (\exists y\, M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof. Then, translate the proof to English.

1. $\forall x\,(T(x) \to M(x))$ ____________________

   2.1. $\exists x\, T(x)$ ____________________

   2.2. $T(c)$ ____________________

   2.3. $T(c) \to M(c)$ ____________________

   2.4. $M(c)$ ____________________

   2.5. $\exists y\, M(y)$ ____________________

2. $(\exists x\, T(x)) \to (\exists y\, M(y))$ ____________________

1.	$\forall x\, T(x) \to M(x)$	Given
2.1.	$\exists x. T(x)$	Assumption
2.2.	$T(c)$	Elim $\exists$: 2.1 (c: specific)
2.3.	$T(c) \implies M(c)$	Elim $\forall$: 1
2.4.	$M(c)$	Modus Ponens: 2.2, 2.3
2.5.	$\exists y\, M(y)$	Intro $\exists$: 2.4
2.	$(\exists x\, T(x)) \to (\exists y\, M(y))$	Direct Proof: 2.1-2.5

English proof:

Suppose that there exists an object c such that $T(c)$. Since we are given that for any object x in the domain, $T(x)$ implies $M(x)$, we know that this must also be true for c. So, we can conclude $M(c)$. This shows that there exists an object y (=c) such that $M(y)$.